

Lévy processes

Suppose we have a process $X = (X_t : t \geq 0)$ which has stationary independent increments. Such a process is called a *Lévy process*, in honour of their creator, the great French probabilist Paul Lévy (1886-1971) [see *Ann. Probab.* 1.1 for his obituary, by Loève]. Then for each $n = 1, 2, \dots$,

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \dots + (X_t - X_{(n-1)t/n})$$

displays X_t as the sum of n independent (by independent increments), identically distributed (by stationary increments) random variables. Consequently, X_t is *infinitely divisible*, so its CF is given by the Lévy-Khintchine formula (L-K). The prime example (anticipating Chapter 3) is:

the Wiener process, or Brownian motion, is a Lévy process.

Poisson Processes.

The increment $N_{t+u} - N_u$ ($t, u \geq 0$) of a Poisson process is the number of failures in $(u, t+u]$ (in the language of renewal theory - see Ch. 1). By the lack-of-memory property of the exponential, this is independent of the failures in $[0, u]$, so the increments of N are *independent*. It is also identically distributed to the number of failures in $[0, t]$, so the increments of N are *stationary*. That is, N has stationary independent increments, so is a Lévy process:

Poisson processes are Lévy processes.

We need an important property: two Poisson processes (on the same filtration) are independent iff they never jump together (a.s.). For proof, see e.g. [R-Y], XII.1.

The Poisson count in an interval of length t is Poisson $P(\lambda t)$ (where the rate λ is the parameter in the exponential $E(\lambda)$ of the renewal-theory viewpoint), and the Poisson counts of disjoint intervals are independent. This extends from intervals to Borel sets:

- (i) For a Borel set B , the Poisson count in B is Poisson $P(\lambda|B|)$, where $|\cdot|$ denotes Lebesgue measure;
- (ii) Poisson counts over disjoint Borel sets are independent.

Poisson (Random) Measures.

If ν is a finite measure, call a random measure ϕ *Poisson* with *intensity* (or characteristic) *measure* ν if for each Borel set B , $\phi(B)$ has a Poisson distribution with parameter $\nu(B)$, and for $B_1, \dots, B_n$, $\phi(B_1), \dots, \phi(B_n)$ are independent. One can extend to σ -finite measures ν : if (E_n) are disjoint

with union $\mathbf{R}$ and each $\nu(E_n) < \infty$, construct ϕ_n from ν restricted to E_n and write ϕ for $\sum \phi_n$.

Poisson Point Processes.

With ν as above a (σ -finite) measure on $\mathbf{R}$, consider the product measure $\mu = \nu \times dt$ on $\mathbf{R} \times [0, \infty)$, and a Poisson measure ϕ on it with intensity μ . Then ϕ has the form

$$\phi = \sum_{t \geq 0} \delta_{(e(t), t)},$$

where the sum is *countable* (for background and details, see [Ber], §0.5, whose treatment we follow here). Thus ϕ is the sum of Dirac measures over ‘Poisson points’ $e(t)$ occurring at Poisson times t . Call $e = (e(t) : t \geq 0)$ a *Poisson point process* with *characteristic measure* ν ,

$$e = Ppp(\nu).$$

For each Borel set B ,

$$N(t, B) := \phi(B \times [0, t]) = \text{card}\{s \leq t : e(s) \in B\}$$

is the *counting process* of B - it counts the Poisson points in B - and is a Poisson process with rate (parameter) $\nu(B)$. All this reverses: starting with an $e = (e(t) : t \geq 0)$ whose counting processes over Borel sets B are Poisson $P(\nu(B))$, then - as no point can contribute to more than one count over disjoint sets, disjoint counting processes never jump together, so are independent by above, and $\phi := \sum_{t \geq 0} \delta_{(e(t), t)}$ is a Poisson measure with intensity $\mu = \nu \times dt$.

Note. The link between point processes and martingales goes back to S. Watanabe in 1964 (Japanese J. Math.). The approach via Poisson point processes is due to K. Itô in 1970 (Proc. 6th Berkeley Symp.); see below, and - in the context of excursion theory - [R-W2], VI §8. For a monograph treatment of Poisson processes, see [Kin].

Lévy Processes and the Lévy-Khintchine Formula.

We can now sketch the close link between the general Lévy process on the one hand and the general infinitely-divisible law given by the Lévy-Khintchine formula (L-K) on the other. We follow [Ber], §1.1.

First, if $X = (X_t)$ is Lévy, the law of each X_1 is infinitely divisible, so given by

$$E \exp\{iuX_t\} = \exp\{-\Psi(u)\} \quad (u \in \mathbf{R})$$

with Ψ a Lévy exponent as in (L-K). Similarly,

$$E \exp\{iuX_t\} = \exp\{-t\Psi(u)\} \quad (u \in \mathbf{R}),$$

for rational t at first and general t by approximation and càdlàg paths. Then Ψ is called the *Lévy exponent*, or *characteristic exponent*, of the Lévy process X .

Conversely, given a Lévy exponent $\Psi(u)$ as in (L-K), construct a Brownian motion (we defer existence and construction to Ch. 3 below), and an independent Poisson point process $\Delta = (\Delta_t : t \geq 0)$ with characteristic measure μ , the Lévy measure in (L-K). Then $X_1(t) := at + \sigma B_t$ has CF

$$E \exp\{iuX_1(t)\} = \exp\{-t\Psi_1(t)\} = \exp\{-t(iau + \frac{1}{2}\sigma^2 u^2)\},$$

giving the non-integral terms in (L-K). For the ‘large’ jumps of Δ , write

$$\Delta_t^{(2)} := \Delta_t \text{ if } |\Delta_t| \geq 1, 0 \text{ else.}$$

Then $\Delta^{(2)}$ is a Poisson point process with characteristic measure $\mu^{(2)}(dx) := I(|x| \geq 1)\mu(dx)$. Since $\int \min(1, |x|^2)\mu(dx) < \infty$, $\mu^{(2)}$ has finite mass, so $\Delta^{(2)}$, a $Ppp(\mu^{(2)})$, is discrete and its counting process

$$X_t^{(2)} := \sum_{s \leq t} \Delta_s^{(2)} \quad (t \geq 0)$$

is compound Poisson, with Lévy exponent

$$\Psi^{(2)}(u) = \int (1 - e^{iux}) I(|x| \geq 1) \mu(dx) = \int (1 - e^{iux}) \mu^{(2)}(dx).$$

There remain the ‘small jumps’,

$$\Delta_t^{(3)} := \Delta_t \text{ if } |\Delta_t| < 1, 0 \text{ else,}$$

a $Ppp(\mu^{(3)})$, where $\mu^{(3)}(dx) = I(|x| < 1)\mu(dx)$, and independent of $\Delta^{(2)}$ because $\Delta^{(2)}$, $\Delta^{(3)}$ are Poisson point processes that never jump together. For each $\epsilon > 0$, the ‘compensated sum of jumps’

$$X_t^{(\epsilon, 3)} := \sum_{s \leq t} I(\epsilon < |\Delta_s| < 1) \Delta_s - t \int x I(\epsilon < |x| < 1) \mu(dx) \quad (t \geq 0)$$

is a Lévy process with Lévy exponent

$$\Psi^{(\epsilon,3)}(u) = \int (1 - e^{iux} + iux)I(\epsilon < |x| < 1)\mu(dx).$$

Use of a suitable maximal inequality allows passage to the limit $\epsilon \downarrow 0$ (going from finite to possibly countably infinite sums of jumps): $X_t^{(\epsilon,3)} \rightarrow X_t^{(3)}$, a Lévy process with Lévy exponent

$$\Psi^{(3)}(u) = \int (1 - e^{iux} + iux)I(|x| < 1)\mu(dx),$$

independent of $X^{(2)}$ and with càdlàg paths. Combining:

THEOREM. For $a \in \mathbf{R}$, $\sigma \geq 0$, $\int \min(1, |x|^2)\mu(dx) < \infty$ and

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iux)I(|x| < 1)\mu(dx),$$

the construction above yields a Lévy process

$$X = X^{(1)} + X^{(2)} + X^{(3)}$$

with Lévy exponent $\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}$. Here the $X^{(i)}$ are independent Lévy processes, with Lévy exponents $\Psi^{(i)}$; $X^{(1)}$ is Gaussian, $X^{(2)}$ is a compound Poisson process with jumps of modulus ≥ 1 ; $X^{(3)}$ is a compensated sum of jumps of modulus < 1 . The jump process $\Delta X = (\Delta X_t : t \geq 0)$ is a $Ppp(\mu)$, and similarly $\Delta X^{(i)}$ is a $Ppp(\mu^{(i)})$ for $i = 2, 3$.

Subordinators.

We resort to complex numbers in the CF $\phi(u) = E(e^{iuX})$ because this always exists - for all real u - unlike the ostensibly simpler moment-generating function (MGF) $M(u) := E(e^{uX})$, which may well diverge for some real u . However, if the random variable X is *non-negative*, then for $s \geq 0$ the *Laplace-Stieltjes transform* (LST)

$$\psi(s) := E(e^{-sX}) \leq E1 = 1$$

always exists. For $X \geq 0$ we have both the CF and the LST to hand, but the LST is usually simpler to handle. We can pass from CF to LST formally by taking $u = is$, and this can be justified by analytic continuation.

Some Lévy processes X have increasing (i.e. non-decreasing) sample paths; these are called *subordinators* ([Ber], Ch. III). From the construction above, subordinators can have no negative jumps, so μ has support in $(0, \infty)$ and no mass on $(-\infty, 0)$. Because increasing functions have BV, one must have paths of (locally) bounded variation, the condition for which can be shown to be

$$\int \min(1, |x|) \mu(dx) < \infty.$$

Thus the Lévy exponent must be of the form

$$\Psi(u) = -idu + \int_0^\infty (1 - e^{iux}) \mu(dx),$$

with $d \geq 0$. It is more convenient to use the Laplace exponent $\Phi(s) = \Psi(is)$

$$E \exp\{-sX_t\} = \exp\{-t\Phi(s)\} \quad (s \geq 0),$$

$$\Phi(s) = ds + \int_0^\infty (1 - e^{-sx}) \mu(dx).$$

Example: The Stable Subordinator. Here $d = 0$, $\Phi(s) = s^\alpha$ ($0 < \alpha < 1$),

$$\mu(dx) = dx / (\Gamma(1 - \alpha)x^{\alpha-1}).$$

The special case $\alpha = 1/2$ is particularly important: this arises as the first-passage time of Brownian motion over positive levels, and gives rise to the Lévy density above.

Classification.

IV (Infinite Variation): The sample paths have infinite variation on finite time-intervals, a.s. This occurs iff

$$\sigma > 0 \text{ or } \int \min(1, |x|) \mu(dx) = \infty.$$

FV (Finite Variation, on finite time-intervals, a.s.):

$$\int \min(1, |x|) \mu(dx) < \infty.$$

IA (Infinite Activity). Here there are infinitely many jumps in finite time-intervals, a.s.: μ has infinite mass, equivalently $\int_{-1}^1 \mu(dx) = \infty$:

$$\nu(\mathbf{R}) = \infty.$$

FA (Finite Activity). Here there are only finitely many jumps in finite time, a.s., and we are in the compound Poisson case:

$$\mu(\mathbf{R}) < \infty.$$

Economic Interpretation.

Suppose X is used as a driving noise process in a financial market model for asset prices (example: $X = BM$ in the Black-Scholes-Merton model). If prices move continuously, the Brownian model is appropriate: among Lévy processes, only Brownian motions have continuous paths ($\mu = 0$, so there are no jumps). If prices move by intermittent jumps, a compound Poisson (FA) model is appropriate - but this is more suitable for modelling economic shocks, or the effects of big transactions. For the more common case of the everyday movement of traded stocks under the competitive effects of supply and demand, numerous small trades predominate, economic agents are price takers and not price makers, and a model with infinite activity (IA) is appropriate.

There is a parallel between the financial situation above - the IA case (lots of small traders) as a limiting case of the FA case (a few large ones) and the applied probability areas of queues and dams. Think of work arriving from the point of view of you, the server. It arrives in large discrete chunks, one with each arriving customer. As long as there is work to be done, you work non-stop to clear it; when no-one is there, you are idle. The limiting situation is that of a dam. Raindrops may be discrete, but one can ignore this from the water-engineering viewpoint. When water is present in the dam, it flows out through the outlet pipe at constant rate (unit rate, say); when the dam is empty, nothing is there to flow out. We will return to these models in Ch. 5; see e.g. [Bin] for details and references.

Lévy Processes as Semi-Martingales.

The Gaussian component $X^{(1)}$ is a martingale; so too is the compensated sum of (small) jumps process $X^{(3)}$, while the sum of large jumps process $X^{(2)}$ is (locally) of bounded variation, being compound Poisson. Thus a Lévy process $X = X^{(1)} + X^{(2)} + X^{(3)}$ is a semimartingale. Indeed, Lévy processes are the prototypes, and motivating examples, of semimings. The natural domain of stochastic integration is (Ch. 1) predictable integrands and seming integrators. Thus, stochastic integration works with a general Lévy process as integrator. Here, however, the theory simplifies considerably. For a monograph treatment of stochastic calculus in this stripped-down setting of Lévy processes, we refer to the forthcoming book [A].

Note: What constitutes pathological behaviour?

Weierstrass, and several other analysts of the 19th C., constructed examples of functions which were continuous but nowhere differentiable. These were long regarded as interesting but pathological. Similarly for the paths of Brownian motion (Ch. 3). This used to be regarded as very interesting mathematically, but of limited relevance to modelling the real world. Then - following the work of B. B. Mandelbrot (plus computer graphics, etc.) - *fractals* attracted huge attention. It was then realised that such properties were typical of fractals, and so - as we now see fractals everywhere (to quote the title of Barnsley's book) - ubiquitous rather than pathological.

The situation with Lévy paths of infinite activity is somewhat analogous. Because one cannot draw them (or even visualise them, perhaps), they used to be regarded as mathematically interesting but clearly idealised so far as modelling of the real world goes. The above economic/financial interpretation has changed all this. 'Lévy finance' is very much alive at the moment (see e.g. [B-K01a,01b,02], [BN-M-R]). Moral: one never quite knows when this sort of thing is going to happen in mathematics!