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Lecture 5

Chapter 3. GAUSSIAN-BASED MODELS

Brownian Motion.

Definition. Brownian motion (BM) on $\mathbf{R}$ is the process $B = (B_t : t \geq 0)$ such that:

(i) $B_0 = 0$;

(ii) B has stationary independent increments (so B is a Lévy process);

(iii) B has Gaussian increments: for $s, t \geq 0$, $B_{t+s} - B_s \sim N(0, t)$;

(iv) B has continuous paths: $t \mapsto B_t$ is continuous ($t \mapsto B(t, \omega)$ is continuous for all $\omega \in \Omega$).

[The path-continuity in (iv) can be relaxed by assuming it only a.s.; we can then get continuity by excluding a suitable null-set from our probability space.]

The fact that Brownian motion so defined *exists* is quite deep, and was first proved by Norbert Wiener (1894-1964) in 1923. In honour of this, Brownian motion is also known as the *Wiener process*, and the probability measure generating it - the measure W on $C[0,1]$ (one can extend to $C[0, \infty)$) by

$$W(A) = P(B_{\cdot} \in A) = P(\{t \mapsto B_t(\omega)\} \in A)$$

for all Borel sets $A \in C[0,1]$ is called *Wiener measure.*

Covariance. Before addressing existence, we first find the covariance function. For $s \leq t$, $B_t = B_s + (B_t - B_s)$, so as $EB_t = 0$,

$$cov(B_s, B_t) = E(B_s B_t) = E(B_s^2) + E[B_s(B_t - B_s)].$$

The last term is $E(B_s)E(B_t - B_s)$ by independent increments, and this is zero, so

$$cov(B_s, B_t) = E(B_s^2) = s \qquad (s \leq t): \qquad cov(B_s, B_t) = \min(s, t).$$

A Gaussian process (one whose finite-dimensional distributions are Gaussian) is specified by its mean function and its covariance function, so among centred (zero-mean) Gaussian processes, the covariance function $\min(s,t)$ serves as the signature of Brownian motion.

Finite-Dimensional Distributions. For $0 \leq t_1 < \ldots < t_n$, the joint law of $X(t_1), X(t_2), \ldots, X(t_n)$ can be obtained from that of $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$. These are jointly Gaussian, hence so are $X(t_1), \ldots, X(t_n)$: the finite-dimensional distributions are *multivariate normal*. Recall that the multivariate normal law in n dimensions, $N_n(\mu, \Sigma)$ is specified by the mean vector μ and the covariance matrix Σ (non-negative definite) by its CF:

$$E \exp\{i\mathbf{u}^T\mathbf{X}\} = \exp\{i\mathbf{u}^T\mathbf{X} - \frac{1}{2}\mathbf{u}^T\Sigma\mathbf{u}\},$$

and when Σ is positive definite (so non-singular), the joint density is given by Edgeworth's theorem:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{1}{2}n}|\Sigma|^{\frac{1}{2}}} . \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}.$$

So to check the finite-dimensional distributions of BM - stationary independent increments with $B_t \sim N(0,t)$ - it suffices to show that they are multivariate normal with mean zero and covariance $cov(B_s, B_t) = \min(s,t)$ as above.

Construction of BM.

It suffices to construct BM for $t \in [0,1])$. This gives $t \in [0,n]$ by dilation, and $t \in [0,\infty)$ by letting $n \to \infty$.

First, take $L^2[0,1]$, and any complete orthonormal system (cons) (ϕ_n) on it. Now L^2 is a Hilbert space, under the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \text{ (or } \textstyle\int fg),$$

so norm $\|f\| := (\int f^2)^{1/2})$. By Parseval's identity,

$$\int_0^1 fg = \sum_{n=0}^{\infty} \langle f, \phi_n\rangle\langle g, \phi_n\rangle$$

(where convergence of the series on the right is in L^2, or in mean square: $\|f - \sum_0^n \langle f, \phi_k\rangle\phi_k\|$
$\to 0$ as $n \to \infty$). Now take, for $s, t \in [0,1]$,

$$f(x) = I_{[0,s]}(x), \qquad g(x) = I_{[0,t]}(x).$$

Parseval's identity becomes

$$\min(s,t) = \sum_{n=0}^{\infty} \int_0^2 \phi_n dx \int_0^t \phi_n(x)dx.$$

Now take (Z_n) independent and identically distributed $N(0,1)$, and write

$$B_t = \sum_{n=0}^{\infty} Z_n \int_0^t \phi_n(x)dx.$$

This is a sum of independent random variables. Kolmogorov's theorem on random series ('three- series theorem') says that it converges a.s. if the sum of the variances converges. This is $\sum_{n=0}^{\infty}(\int_0^t \phi_n(x)dx)^2$, $= t$ by above. So the series above converges a.s., and by excluding the exceptional null set from our probability space (as we may), everywhere.

The Haar System. Define

$$H(t) = 1 \text{ on } [0, \tfrac{1}{2}), \qquad -1 \text{ on } [\tfrac{1}{2}, 1], \qquad 0 \text{ else.}$$

Write $H_0(t) \equiv 1$, and for $n \geq 1$, express n in dyadic form as $n = 2^j + k$ for a unique $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2_j - 1$. Using this notation for n, j, k throughout, write

$$H_n(t) := 2^{j/2} H(2^j t - k)$$

(so H_n has support $[k/2^j, (k+1)/2^j]$). So if m, n have the same j, $H_m H_n \equiv 0$, while if m, n have different js, one can check that $H_m H_n$ is $2^{(j_1+j_2)/2}$ on half its support, $-2^{(j_1+j_2)/2}$ on the other half, so $\int H_m H_n = 0$. Also H_n^2 is 2^j on $[k/2^j, (k+1)/2^j]$, so $\int H_n^2 = 1$. Combining:

$$\int H_m H_n = \delta_{mn},$$

and (H_n) form an orthonormal system, called the *Haar system.* For completeness: the indicator of any dyadic interval $[k/2^j, (k+1)/2^j]$ is in the linear span of the H_n (difference two consecutive H_ns and scale). Linear combinations of such indicators are dense in $L^2[0,1]$. Combining: the Haar system (H_n) is a cons in $L^2[0,1]$.

The Schauder System.

We obtain the *Schauder system* by integrating the Haar system. Consider the triangular function (or 'tent function')

$$\Delta(t) := 2t \qquad (0 \leq t \leq \frac{1}{2}), \qquad 2(1-t) \qquad (\frac{1}{2} \leq t \leq 1), \qquad 0 \qquad \text{else.}$$

Write $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$, and define the nth *Schauder function* Δ_n by

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k \geq 1).$$

Note that Δ_n has support $[k/2^j, (k+1)/2^j]$ (so is 'localized' on this dyadic interval, which is small for n, j large). We see that

$$\int_0^t H(u)du = \frac{1}{2}\Delta(t),$$

and similarly

$$\int_0^t H_n(u)du = \lambda_n \Delta_n(t),$$

where $\lambda_0 = 1$ and for $n \geq 1$,

$$\lambda_n = \frac{1}{2}.2^{-j/2} \qquad (n = 2^j + k \geq 1).$$

The Schauder system (Δ_n) is again a cons on $L^2[0,1]$.

THEOREM (Paley-Wiener-Zygmund, 1932. For $(Z_n)_0^\infty$ independent $N(0,1)$ random variables, λ_n, Δ_n as above,

$$B_t := \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)$$

converges uniformly on $[0,1]$, a.s. The process $B = (B_t : t \in [0,1])$ is Brownian motion.

LEMMA. For Z_n independent $N(0,1)$,

$$|Z_n| \leq C\sqrt{\log n} \qquad \forall n \geq 2,$$

for some random variable $C < \infty$ a.s.

Proof. For $x > 1$,

$$P(|Z_n| \geq x) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}u^2} du \leq \sqrt{2/\pi} \int_x^\infty u e^{-\frac{1}{2}u^2} du = \sqrt{2/\pi} e^{-\frac{1}{2}x^2}.$$

So for any $a > 1$,

$$P(|Z_n| > \sqrt{2a \log n)} \leq \sqrt{2/\pi} \exp(-a \log n) = \sqrt{2/\pi}.n^{-a}.$$

Since $\sum n^{-a} < \infty$ for $a > 1$, the Borel-Cantelli lemma gives

$$P(|Z_n| > \sqrt{2a \log n} \text{ for infinitely many } n) = 0.$$

So

$$C := \sup_{n \geq 2} \frac{|Z_n|}{\sqrt{\log n}} < \infty \qquad a.s.$$

Proof of the Theorem.
1. Convergence. Choose J and $M \geq 2^J$; then

$$\sum_{n=M}^{\infty} \lambda_n |Z_n| \Delta_n(t) \leq C \sum_{M}^{\infty} \lambda_n \sqrt{\log n} \Delta_n(t).$$

The right is majorized by

$$C. \sum_{J}^{\infty} \sum_{k=0}^{2^j - 1} \frac{1}{2} . 2^{-j/2} \sqrt{j+1} \Delta_{2^j + k}(t)$$

(perhaps including some extra terms at the beginning, using $n = 2^j + k < 2^{j+1}$, $\log n \leq (j+1) \log 2$, and $\Delta_n(.) \geq 0$, so the series is absolutely convergent). In the inner sum, only one term is non-zero (t can belong to only one dyadic interval $[k/2^j, (k+1)/2^j)$), and each $\Delta_n(t) \in [0,1]$. So

$$LHS \leq C \sum_{j=J}^{\infty} \frac{1}{2} . 2^{-j/2} \sqrt{j+1} \qquad \forall t \in [0,1],$$

and this tends to 0 as $J \to \infty$, so as $M \to \infty$. So the series $\sum \lambda_n Z_n \Delta_n(t)$ is absolutely and uniformly convergent, a.s. Since continuity is preserved under uniform convergence and each $\Delta_n(t)$ (so each partial sum) is continuous, B_t is continuous in t.
2. Covariance. By absolute convergence and Fubini's theorem,

$$EB_t = E \sum_{0}^{\infty} \lambda_n Z_n \Delta_n(t) = \sum \lambda_n \Delta_n(t) . EZ_n = \sum 0 = 0.$$

So the covariance is

$$E(B_s B_t) = E[\sum_m Z_m \int_0^s \phi_m . \sum_n Z_n \int_0^t \phi_n] = \sum_m \sum_n E[Z_m Z_n] \int_0^s \phi_m \int_0^t \phi_n,$$

or as $E[Z_m Z_n] = \delta_{mn}$,

$$\sum_n \int_0^s \phi_m \int_0^t \phi_n = \min(s,t),$$

by the Parseval calculation above.

3. Joint Distributions. Take $t_1, \ldots, t_m \in [0,1]$, we have to show that $(B(t_1), \ldots, B(t_n))$ is multivariate normal, with mean vector 0 and covariance matrix $(\min(t_i, t_j))$. The multivariate CF is

$$E\exp\{i\sum_{j=1}^{m} u_j B(t_j)\} = E\exp\{i\sum_{j=1}^{m} u_j \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)\},$$

which by independence of the Z_n is

$$\prod_{n=0}^{\infty} E\exp\{i\lambda_n Z_n \sum_{j=1}^{m} u_j \Delta_n(t_j)\}.$$

Since each Z_n is $N(0,1)$, the RHS is

$$\prod_{n=0}^{\infty} \exp\{-\frac{1}{2}\lambda_n^2 (\sum_{j=1}^{m} u_j \Delta_n(t_j))^2\} = \exp\{-\frac{1}{2}\sum_{n=0}^{\infty} \lambda_n^2 (\sum_{j=1}^{m} u_j \Delta_n(t))^2\}.$$

The sum in the exponent on the right is

$$\sum_{n=0}^{\infty} \lambda_n^2 \sum_{j=1}^{m}\sum_{k=1}^{m} u_j u_k \Delta_n(t_j)\Delta_n(t_k) = \sum_{j=1}^{m}\sum_{k=1}^{m} u_j u_k \sum_{n=0}^{\infty} \int_0^{t_j} H_n(u)du. \int_0^{t_k} H_n(u)du,$$

giving

$$\sum_{j=1}^{m}\sum_{k=1}^{m} u_j u_k \min(t_j, t_k),$$

by the Parseval calculation, as (H_n) are cons. Combining,

$$E\exp\{i\sum_{j=1}^{m} u_j B(t_j)\} = \exp\{-\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} u_j u_k \min(t_j, t_k)\}.$$

This says that $(B(t_1), \ldots, B(t_n))$ is multinormal with mean 0 and covariance function $\min(t_j, t_k)$ as required. This completes the construction of BM.

Wavelets. The Haar system (H_n), and the Schauder system (Δ_n) obtained by integration from it, are examples of *wavelet systems*. The original function, H or Δ, is a *mother wavelet*, and the 'daughter wavelets' are obtained from it by dilation and translation. The expansion of the Theorem is the *wavelet expansion of BM* with respect to the Schauder system (Δ_n). For any $f \in C[0,1]$, we can form its wavelet expansion

$$f(t) = \sum_{n=0}^{\infty} c_n \Delta_n(t),$$

with *wavelet coefficients* c_n. Here c_n are given by

$$c_n = f(\frac{k+\frac{1}{2}}{2^j}) - \frac{1}{2}[f(\frac{k}{2^j}) + f(\frac{k+1}{2^j})].$$

This is the form that gives the $\Delta_n(.)$ term its correct triangular influence, localized on the dyadic interval $[k/2^j, (k+1)/2^j]$. Thus for f BM, $c_n = \lambda_n Z_n$, with λ_n, Z_n as above. The wavelet construction of BM above is, in modern language, the classical 'broken-line' construction of BM due to Lévy in his book of 1948 - the *Lévy representation* of BM using the Schauder system, and extended to general cons by Cieselski in 1961; see [McK], §1.2 for a textbook account. The account above is from [Ste]. The earliest expansion of BM - 'Fourier-Wiener expansion' - used the trigonometric cons (Paley & Zygmund 1930-32, Paley, Wiener & Zygmund 1932; see [Kah], Preface and §16.3.

Note. We shall see that Brownian motion is a *fractal*, and wavelets are a useful tool for the analysis of fractals more generally. For background, see e.g. [Hol], §4.4.