MULTIVARIATE ELLIPTIC PROCESSES

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I. Static picture: Distribution theory
II. Dynamic picture: Lévy processes
III. Dynamic picture: Diffusions

I. STATIC PICTURE: DISTRIBUTION THEORY

References:

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BS N. H. BINGHAM and R. SCHMIDT: Distributional and temporal dependence structure of high-frequency financial data: A copula approach. *From stochastic analysis to mathematical finance: The Shiryaev Festschrift*, ed. Yu. Kabanov, R. Liptser & J. Stoyanov) 69-92, Springer, 2006.

Recall the bench-mark Black-Scholes(-Merton) model. The evolution of a stock price S_t is modelled by a stochastic differential equation (SDE)

$$dS_t = S_t.(\mu dt + \sigma dB_t), \qquad (SDE)$$

where μ is the mean growth rate, σ is the volatility and $B = (B_t)$ is Brownian motion (BM). The solution is

$$S_t = \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

In particular, S_t has a *log-normal distribution*. Similarly in higher dimensions. In d dimensions, μ is a d-vector, B_t is d-dimensional BM, S_t is a d-dimensional stochastic process with log-normal components, and σ is a $d \times d$ matrix. So log-prices are *multivariate normal* or Gaussian. Recall that this distribution has characteristic function (CF)

$$M(t) = \exp\{it^T\mu - \frac{1}{2}t^T\Sigma t\},$$

where μ is the mean vector and Σ is the covariance matrix, and density

$$f(x) = const. \exp\{-\frac{1}{2}(t-\mu)^T \Sigma^{-1} (t-\mu)\}.$$

(Edgeworth's formula, 1893).

Markowitz Theory.

Recall the two key insights of Markowitz's thesis of 1952:

1. Look at risk (σ, Σ) and return (μ) together not separately (*mean-variance theory*).
2. *Diversify*. Hold a (large) number d of assets, and *balance* your portfolio by choosing a range of assets with negative correlations.

Problems

The Black-Scholes model gives Gaussian log-prices, which have *ultra-thin tails*. Real financial data typically have *much fatter tails*.

One thus seeks a model which retains (as much as possible) the mathematical tractability of the Gaussian model but is not restricted to ultra-thin Gaussian tails. One way to do this is to use an *elliptically contoured* model, where the density is of the form (generalizing Edgeworth's formula)

$$f(x) = g(Q), \qquad Q := (x - \mu)^T \Sigma^{-1} (x - \mu),$$

where g is a positive function of a positive variable (the *density generator*). We summarize this as

$$f \in EC, \qquad \text{or} \qquad f \in EC_d(\mu, \Sigma, g).$$

Then if X is a random d-vector with law of type EC, one has a stochastic representation

$$X - \mu = RA^T U, \qquad (SR)$$

where Σ has Cholesky decomposition

$$\Sigma = A^T A,$$

U is uniformly distributed over the unit sphere in d dimensions, and $R > 0$ is a random variable.

Examples.
1. Gaussian: here $g(x) = const.e^{-\frac{1}{2}x}$. The tails are ultra-thin, as above. Suitable for modelling, say, monthly returns.
2. Student t in d dimensions with n degrees of freedom: here $g(x) = const.(1+\frac{x^T\Sigma^{-1}x}{n})^{-\frac{1}{2}(n+d)}$. Heavy tails – decay like a power. May be useful for modelling, say, high-frequency returns. Can be obtained as a *normal variance mixture* (NVM) – Gaussian with mean 0 and covariance matrix $u\Sigma$, where u is random with inverse Gamma distribution, IG ('mixing law IG').
3. Generalized hyperbolic, GH: again NVM, with mixing law generalized inverse Gaussian (GIG). Semi-heavy tails (log-tails decay linearly). Suitable for modelling daily returns, say.
Note. 1. Observe how varying the return interval can alter the character of the return distribution!
2. Returns correspond to discrete time; log-prices are more suitable for continuous time.
3. Data are discrete; theory may be easier in continuous time (we return to this later).

Infinite divisibility and Lévy processes

A random variable X with CF ϕ is called *infinitely divisible* (ID) if for each $n = 1, 2, \ldots$ $X = X_1 + \ldots + X_n$ with $X_1, \ldots, X_n$ independent and identically distributed (iid) – equivalently, $\phi = {\phi_n}^n$ for some CF ϕ_n. The ID laws are given by the *Lévy-Khintchine formula*, in terms of a triple (a, σ, ν), where a is real (the drift), $\sigma \geq 0$ (the Gaussian component), and ν is a measure (the Lévy measure) satisfying an integrability condition. An ID law corresponds to a *Lévy processes* $(X_t)_{t \geq 0}$ – stochastic process (SP) with stationary independent increments – by $X \leftrightarrow X_1$.

Examples.

1. Brownian motion [normal or Gaussian distributions].
2. Poisson process [Poisson distribution].
3. Student t processes [Student t distributions].
4. Generalized hyperbolic processes [generalised hyperbolic distributions, GH].

Self-decomposability (SD).
Call a random variable X *self-decomposable* (SD) if for each $c \in (0, 1)$,

$$X =_d cX +_{ind} X_c$$

for some r.v. X_c (note the similarity to $AR(1)$!) – equivalently, the CF ϕ satisfies $\phi(t) = \phi(ct).\phi_c(t)$ for some CF ϕ_c. Then SD laws are ID:

$$SD \subset ID,$$

and the SD laws are known in terms of the Lévy-Khintchine formula.
Examples: Gaussian, Student t and GH laws are SD.
Type G.
Suppose now that

$$Y = \sigma\epsilon,$$

where

$$\epsilon \sim N_d(0, \Sigma)$$

is a random d-vector, multivariate normal (Gaussian) with mean 0 and covariance matrix Σ and σ is independent of ϵ with σ^2 ID. Then Y is said to be of *type* G (M. B. Marcus, 1987). Then Y has CF

$$\psi_Y(t) = \phi(\frac{1}{2}t^T \Sigma t),$$

where ϕ is the Laplace-Stieltjes transform (LST) of σ^2. Then

$$X := Y + \mu$$

is elliptically contoured:

$$X \sim EC_d(\mu, \Sigma, \phi)$$

say. We specialize further from σ^2 ID to σ^2 SD. Then (check) X above is also SD.

The message of **BK**, **BKS** is that this setting provides a very suitable and flexible way of modelling return or log-price laws in many dimensions.

Note. The concept of type G is not made explicit in **BK**, **BKS**.

II. DYNAMIC PICTURE: LÉVY PROCESSES

Risk driver.

We now take a dynamic version of (SR):

$$X_t - \mu = R_t A^T U_t, \qquad (SR_t)$$

where $X = (X_t)_{t \geq 0}$ is a d-dimensional SP, $R = (R_t)$ is a SP on the positive half-line, and $U = (U_t)$ is BM on the surface of the unit sphere in d dimensions. We call X a *multivariate elliptical process* (MEP) with *risk driver* R.

Interpretation: X is the log-price process of our portfolio of d assets. We need (μ, Σ) (mean vector, covariance matrix) as a parameter, by Markowitz. We assume here that the variability can be adequately modelled by a *one-dimensional* driving noise process, the *risk driver* R, which represents the uncertainty in the economic environment. This greatly simplifies computations, and avoids the *curse of dimensionality*. From (SR_t):

$$var(X_t|R_t) = R_t^2 \Sigma, \ var(X_t) = E[R_t^2]\Sigma. \quad (vol)$$

This gives *stochastic volatility*, one of the "stylized facts" of mathematical finance. Large or small values of R tend to be followed by large or small values for R, so for the covariance matrix, or *volatility matrix*, giving *volatility clustering*, another stylized fact.

Processes of Ornstein-Uhlenbeck type.

Recall the classical Ornstein-Uhlenbeck process, given by the SDE

$$dV_t = -cV_t + \sigma dB_t.$$

We generalize this as follows:

$$dY_t = -cY_t + dZ_t. \qquad (OU)$$

Here $c > 0$, and $Z = (Z_t)$ is a positive Lévy process (subordinator), called the background driving Lévy process (BDLP). The solution to SDE (OU) is a process of *Ornstein-Uhlenbeck type*. We quote:

(a) If Z is a BDLP whose Lévy measure ν satisfies the log-integrability condition

$$\int \log^+(|x|) d\nu(x) < \infty, \qquad (logint)$$

then (OU) has a unique strong solution $Y = (Y_t)$, with an SD limit law Y_∞.
(b) Conversely, every SD law is the limit law of a process of OU type.
1. *Log-prices*.
The above gives a model for the log-prices of a d-dimensional portfolio, which has two desirable properties:
(a) the dynamics are driven by a *one-dimensional* noise process, the risk driver R;
(b) the process is stationary, and settles down to equilibrium.
2. *Stochastic Volatility (SV) continued*.
Barndorff-Nielsen and Shephard (JRSS B 2001) introduce a SV model of this type:

$$dy_t = (\mu + c\sigma_t^2)dt + \sigma_t dB_t, \quad d\sigma_t^2 = -\lambda\sigma_t^2 dt + dz_t,$$

where $y = (y_t)$ is the log-price process and the BDLP $z = (z_t)$ is a subordinator (i.e. $z_t > 0$, which ensures the volatility $\sigma_t^2 > 0$ also).

III. DYNAMIC PICTURE: DIFFUSIONS.

In II above, the (log-)price process has *jumps* – the only Lévy process without jumps is Brownian motion, which takes us back to the Black-Scholes model. Typically, jump processes will give a market model which is *incomplete*, in contrast to the Black-Scholes model, which is complete. This section describes an alternative to the above Lévy-based models with jumps, based instead on *diffusions*.

In the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \qquad (diff)$$

subject to suitable conditions on the drift $b(.)$ and diffusion coefficient $\sigma(.)$, the SDE (diff) has a unique strong solution, which is path-continuous and strong Markov, i.e. a *diffusion* process. It may also be described by the differential operator

$$bD + \frac{1}{2}\sigma^2 D^2, \qquad D := d/dx,$$

(together with boundary conditions at end-points). We shall take the risk-driver $R = (R_t)$ as a positive diffusion, and impose the boundary condition that 0 is a reflecting boundary. The diffusion has a *speed measure* and a *scale function*. We take the speed measure finite, so it can be normalized to a probability measure on $(0, \infty)$. We take this absolutely continuous, with density f say. Then
(i) the diffusion is *ergodic* – it has a limit distribution as $t \to \infty$,
(ii) this limit distribution has density f,
(iii) the process is time-reversible (from the boundary condition – 0 is reflecting).
The density f is given by the DE

$$D(\sigma^2 f) = 2bf. \qquad (DE)$$

We shall take the function $\sigma(.)$ as known [because if we could observe the path exactly, we could find its quadratic variation and get $\sigma(.)$ from that; various approximation results mean that we can approximate this in practice]. So we can specify f and find b from (DE), or vice versa.

Example. 1. *Gamma diffusion*. Here f has the Gamma distribution $\Gamma(\alpha, \nu)$ ($\alpha > 0$, $\nu > 0$),

$$f(x) = \frac{\alpha^\nu}{\Gamma(\nu)}.x^{\nu-1}e^{-\alpha x}, \qquad (x > 0).$$

We take σ constant. Then

$$b(x) = \frac{1}{2}\sigma^2.(\frac{\nu-1}{x} - \alpha).$$

2. *Heston or Cox-Ingersoll-Ross (CIR) model*. Here $\sigma(x) = c\sqrt{x}$.

Note. Motivated as here by financial modelling, there has been much recent work on statistical estimation for diffusions. See e.g. the book

Yu. A. KUTOYANTS: *Statistical estimation for ergodic diffusions*, Springer, 2004,

and many papers in the journal *Statistical Inference for Stochastic Processes* (SISP).

Our approach applies all this in many dimensions.

DISCRETE v. CONTINUOUS TIME

Is time discrete or continuous? Should we use discrete-time or continuous-time models in mathematical finance? The answer is that we need *both*.

In favour of discrete time: (a) data is discrete; (b) much of econometrics – e.g., GARCH models – is in discrete time.

For current work here in discrete time, see

SS Rafael SCHMIDT and Christian SCHMIEDER: *Modelling dynamic portfolio risk using risk drivers of elliptical processes*. Preprint, Dept. Economics, U. Cologne [rafael.schmidt@uni-koeln.de].

In favour of continuous time: theory works more smoothly – e.g., Itô calculus, Lévy processes, diffusions.

Much current work is devoted to extending ARMA and GARCH methods to continuous time (CARMA and COGARCH): see recent papers of P. J. BROCKWELL, Alexander LINDNER,Vicky FASEN and others.

For econometrics in continuous time, see

A. R. (Rex) BERGSTROM (1925-2005).