15 Décembre 2011: Journée Paul Lévy

Inauguration of the Salle Paul Lévy,

Salle 113, Aile 16-26, U. Paris-VI, Pl. Jussieu

THE WORLDWIDE INFLUENCE OF PAUL LÉVY IN THE DOMAINS OF PROBABILITY

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Sources

M. Loève, Paul Lévy 1886-1971, *Annals of Probability* **1**.1 (1973), 1-18.
[CP] P. Lévy, *Calcul des Probabilités*, GV, 1925;
[TAVA] P. Lévy, *Théorie de l'addition des variables aléatoires*, G-V, 1937/1954;
[PSMB] P. Lévy, *Processus stochastiques et mouvement brownien*, G-V, 1948/1965,
[QAPM] P. Lévy, *Quelques aspects de la pensée d'un mathématicien*, Blanchard, 1970
Marc Barbut & Laurent Mazliak, Lévy's three lectures of 1919 on probability, *EJHPS* **4**.1 (2008);
Laurent Mazliak, How Paul Lévy saw Jean Ville and martingales, *EJHPS* **5**.1 (2009);
Laurent Mazliak, The ghosts of the Ecole Normale: Life, death and legacy of René Gateaux. *Historia Mathematica*, 2012.
NHB, Measure into probability: from Lebesgue to Kolmogorov. *Biometrika*, 2000.
NHB, Jozef Marcinkiewicz: Life and work. To appear.

Life

Born Paris, 1886; son and grandson of mathematicians; died Paris 1971, aged 85

Brilliant all-round student; Lycée Saint Louis; Ecole Polytéchnique (graduated first, 1905; first paper).

Student of Hadamard, doctoral thesis 1912; Profesor, Ecole des Mines, 1913

Military service 1906-7 and WWI, 1914-18, artillery. (No doubt Lévy was in the artillery because of his mathematical talent; perhaps also his work there laid some foundation for his later work on probability. There are other famous mathematical gunners: Napolean, Littlewood, Linnik, Hammersley, ... (and rocketeers – D. G. Kendall, Rankin, Rosser, ...)

1920-59, Professor of Analysis, Ecole Polytéchnique

1964, elected to the Académie des Sciences

10 books and over 270 papers, over 150 on probability theory

Early work: Functional analysis
First 40 papers, 1905-20, on analysis
Asked by Hadamard to prepare the posthumous papers of René Gateaux (1889-1914), also a pupil of Hadamard, who had visited Volterra, for publication.
Functional differentiation (Gateaux, Hadamard, Fréchet, ...); functional calculus (spectral theory; Dunford & Schwartz; Sz.-Nagy & Foias, ...)
Measures in function space (Wiener measure, 1923, following work by Daniell, Gateaux; Lévy meets Wiener, 1922)
Books on functional analysis, 1922, 1925, 1951; *Cours d'analyse* **1** (1929), **2** (1930). Volterra's work; Green functions. Not cited in Banach (1932), von Neumann (1932) or Courant & Hilbert (1931, 1937); cited in Dunford & Schwartz I (1958), II (1963); cited extensively in the 1927 book of Vito Volterra (1860-1940); Engl. tr. 1929, Dover, 1959.
Functions of regular growth: 1926ab, 1927, 1928abc.

Early work on probability: the Gaussian law

1919: asked to give three lectures at the Ecole Polytéchnique, on "notions of Calculus of Probabilities and the role of the Gaussian law in the theory of errors". These changed both Lévy's life and Probability Theory.

Theme of Lévy's early work on probability (and much of his later work): what leads to a Gaussian law? Intuitive answer: Gaussianity is the signature of an effect with a large number of individually negligible causes.

Subsidiary theme: what leads to a Poisson law? Intuitive answer: Poissonianity is the signature of shocks appearing 'out of the blue'.

Stable laws (1919, arising from his lectures): beyond the Gaussian – Cauchy, followed by the whole class, in terms of CFs.

[CP]: First textbook exposition of probability theory written by a first-rate mathematician in the age of measure theory.

First rigorous textbook exposition of characteristic functions (CFs).

Lévy continuity theorem for CFs: bicontinuous bijection between the classes of CFs (under topology of uniform convergence on compacts) and distribution functions (under the *Lévy metric*) (improved by Bochner in 1933: continuity of the limit of CFs at 0 is enough for 'lim CFs = CF lim').
Types of pr. laws (corresponding to the Lebesgue decomposition: 'd, ac, s').
Modes of convergence.

Middle period on probability
Continued fractions: 1929, 1930, 1936, 1952; TAVA Ch. IX
Concentration functions
Lévy symmetrization inequalities
Martingales, 1934 on (the name is due to Ville, 1939; cf. Mazliak, 2009)
Lévy zero-one law, the first martingale convergence theorem, 1934, 1935; TAVA, Ch. VIII;

modern textbook accounts are in Doob, 7.5, Chung, Th. 9.4.8, Kallenberg, Th. 7.23
Arithmetic of probability laws (convolution factors of Gaussians are Gaussian, Poissons are Poisson, ...): 1937, 1938
[TAVA], 1937: Continuity theorem (III, Th. 17.1,2)
Lévy's equivalence theorem for random series:for series of independent terms, convergence a.s., in pr. and in dn. are equivalent (VI, T. 44)
Lévy-Khintchine formula, Ch. VII; de Finetti 1929, 1930; Kolmogorov 1932; Lévy 1934; Khintchine 1936.
Lévy processes have benefitted greatly from having this name – succinct, and apt. They were originally stochastic processes with stationary increments/additive processes/differential processes/decomposable processes/PAIS/... The name is used by Fristedt in his survey of 1974, and stuck (I believe it is due to Orey). Loève, Lévy's obituarist, uses decomposable, even in

the 4th edition of his book (1978). The standard modern works are Bertoin 1996 and Sato 1999; also Applebaum 2004/09 (stochastic calculus aspects); all use 'Lévy process'.

Late work on probability
Lévy pioneered the study in fine detail of the (sample) paths of Brownian motion (and indeed, stochastic processes generally).
[PSMB], 1948/1965 (1st ed. pre-Doob; early chapters on generalities): Lévy's "broken-line" construction of BM (a wavelet expansion, in modern language).
Ch. IV, stationary processes in continuous time (Loève, ... – relevant to prediction theory).
Ch. VI, Étude approfondie du mouvement brownien linéaire. Brownian local time (below); quadratic variation of Brownian paths; Lévy's martingale characterization of BM (as the continuous mg with QV t); Lévy's modulus of continuity of

Brownian paths; Lévy's law of the iterated logarithm for BM.
Ch. VII, Le mouvement brownien plan. BM and analytic functions (below).
Ch. VII, Le mouvement brownien à plusieurs paramètres. Random fields – e.g., spatial (or spatio-temporal) processes; processes parametrised by spheres or other manifolds, etc.
Lévy-Schoenberg kernels (R. Gangolli, 1964 *Acta Math.*, 1967 4th BS; NHB, 1970s).
Ch. III, Compléments, 1965: Determinism of BM in Hilbert space: BM in a ball determines BM everywhere (extreme irregularity, in contrast to Complex Analysis).

Other themes
Obituaries etc.: Gateaux 1919, Liouville 1931, Doeblin 1955, Hadamard 1967
Logic: 1926, 1927, 1930; Zermelo 1950
Markov chains: 1951; Feller-McKean chains 1959; Semi-Markov processes 1953.
Wiman-Valiron method in Complex Analysis 1925; Summability, 1926

LÉVY'S IMPACT

1. Lévy as pioneer of paths.
The Lévy-Khintchine formula giving the general id CF (a distributional, or static, result) is best seen in the context of the *Lévy-Itô decomposition* of the *paths* of a Lévy process, decomposing them into the sum of the continuous part, the large jumps (compound Poisson) and the compensated sum of small jumps. This was done by Itô in 1942, and again much later (and more efficiently) using excursion theory in 1972. The idea of working directly with paths has permeated modern probability (see in particular the works of David Williams).

2. Lévy and local time
Lévy's PSMB contains the following result on Brownian motion $B = (B_t)$ and its supremum process $\overline{B}$ ($\overline{B}_t := \sup_{[0,t]} B(.)$):

$$(\overline{B} - B, \overline{B}) =_d (|B|, L).$$

Here $|B|$ vanishes only on the zero-set Z of B and L grows only on Z, while $\overline{B} - B$ vanishes only on the record-set R (where B is at a current record – equal to its supremum to date) and $\overline{B}$ grows only on R. This process L (Lévy's 'mesure de voisinage') is called the *local time* of B. This concept is one of Lévy's outstanding achievements. Using it, one can extend Itô's formula in stochastic calculus for $f(B)$ to the *Itô-Tanaka formula*, valid for $f \notin C^2$ (e.g., f convex). It is needed to describe *reflected* Brownian motion (on hitting a boundary – or boundary surface, in higher dimensions). Local time can be defined for more general processes (see e.g. Azéma-Yor, *Temps locaux*, Astérisque 52-53 (1978) for semi-martingales). There is a theory of *Gaussian* local times, due to S. M. Berman (1970s: the rougher the paths of the process the smoother those of its local time, and vice versa, with BM the 'ideal balance' between the two).

3. Lévy, Ville and martingales

Lévy's contributions to martingale theory are profound, but are complicated both by his not using the term for many years and by his unduly negative view of the work of Jean-André Ville (1910-89), who introduced the term. See Mazliak (2009) for a detailed account. Lévy (in Ch. VIII of TAVA) used martingale-like conditions to extend the convergence results for random series from independent to dependent terms. An example is Lévy's zero-one law (from his 1936 JMPA paper), which Loève called the first martingale convergence theorem. It is related both to Kolmogorov's zero-one law, and to the later convergence theorem for *uniformly integrable* (UI) martingales.

Martingale theory was also studied by the Polish school (Marcinkiewicz, Zygmund, ...), systematised in the famous Chapter VII in Doob's *Stochastic processes* of 1953, and is now ubiquitous, thanks to the later work of many people: Burkholder, Gundy, ...

4. Lévy processes as prototypes of semimartingales

Meyer introduced the concept of semimartingales (processes decomposable as a sum of a local martingale and a finite-variation process) as providing a suitable class of integrators for stochastic integrals. Meyer states (*Un cours sure les intégrales stochastiques*, 255-6; Sém. Prob. X, LNM 511 (1976) – of the limit of compensated small jumps existing) that "this idea of Paul Lévy can be followed throughout the course". Semi-martingales (Dellacherie-Meyer Vol. 2 (1980), VII.2) are now known to be the most general class of integrators (together with predictable integrands) with good properties (Bichteler).

5. Lévy and complex analysis

Lévy's result [PSMB] that if Z is planar BM and f is holomorphic, $f(Z)$ is planar BM with a

time change led to extensive cross-fertilisation between Complex Analysis and martingales in general and BM in particular. See e.g. Getoor and Sharpe, conformal martingales, 1972; B. J. Davis (*Ann. Prob.* **7**.6 (1979), 913-932; Picard's theorem (Davis, TAMS 1975); the Burkholder-Gundy-Silverstein theorem on H_p spaces; and the book
R. DURRETT: *Martingales and Brownian motion in analysis*, Wadsworth, 1984.

6. Lévy-Itô decomposition; Itô excursion theory

This is touched on in 1 above. Excursion theory proper stems from E. B. Dynkin, TPA, 1968.

7. Lévy as pioneer of fractals
Lévy began the study in fine detail of Brownian paths, in one or higher dimensions, in PSMB. He used Hausdorff measure to study Brownian paths in 1953 (a theme developed later by Taylor, Hawkes, ...). Because of Brownian scaling (or self-similarity), Brownian paths are *fractals*. Hence so too are small random sets such as Z and R above. Fractals are rough: Brownian paths are a.s. non-differentiable a.e. (Paley-Wiener-Zygmund 1933); non-increasing a.e. (Dvoretsky-Erdös-Kakutani, 4th BS 1961); ...
Exceptional points on Brownian paths have been studied in great detail (double and multiple points; fast and slow points; ...).
B. B. Mandelbrot (1924-2010) popularised the area (partly with computer graphics), and gave it a memorable name – which helps!

8. Wiener-Lévy theorem; Banach algebras
Wiener's generalised harmonic analysis and Tauberian theorem (1930, 1932) included the statement that if a function f has an absolutely

convergent Fourier series, then in any region where f does not vanish $1/f$ also has an absolutely convergent Fourier series. Lévy (1934) generalised this by replacing the reciprocal by a more general analytic function.
One of the first convincing triumphs of modern (functional) rather than classical analysis was (Gelfand, Godement and others) to extend this theory to the context of Banach algebras and abstract harmonic analysis (whence the Dunford and Schwartz citations above).

9. Lévy processes in mathematical finance.
Benchmark model: Black-Scholes(-Merton) model, 1973; driving noise Brownian motion. Lévy processes can model jumps, and have fatter tails.

Lévy the person

I didn't know Lévy. I know him only through his written work (including his autobiography), anecdotes from those who knew him, and historical writings (esp. Barbut-Mazliak 2008, Mazliak 2009, 2012).

Lévy, by his own admission [QAPM], didn't read the works of other mathematicians much. This has its costs. (By contrast, Fréchet, one of the great mathematicians of the 20th C, read everything.)

Lévy was 'a scathing person' (Mazliak, 2009, p.15: 'acerbe, mordant, cinglant, caustique'). Once he had formed a negative view of another person's work, he tended to stick to it. This had negative consequences for the careers of both Louis Bachelier (1870-1946), the father of mathematical finance (from his 1900 thesis *Théorie de la spéculation*), and Jean Ville (above).

In his autobiography (p.67-68) Lévy writes

poignantly of realising too late, after first seeing Kolmogorov's pioneering *Grundbegriffe* of 1933, the opportunity of writing such a book that he himself had missed.
Lévy was elected an Honorary Member of the London Mathematical Society in 1963. He was elected to the Academy of Sciences in 1964, aged 78 (background in Barbut-Locker-Mazliak, *Paul Lévy– Maurice Fréchet: 50 ans de correspondance mathématique*, 2004.
I have myself heard senior French mathematicians saying that the mathematical establishment was ashamed of Lévy (presumably because his discursive writing style was felt to lack elegance or rigour), and that his Jewishness drew some adverse comment; perhaps the above negativity may be relevant here.
Lévy was the father-in-law of the great analyst Laurent Schwartz (1915-2002). It is often said that mathematical talent descends from father to son-in-law (cf. also Edmund Landau and I.

J. Schoenberg, Sir Ronald Fisher and George Box, Paul Malliavin and Anton Thalmaier, ...). (Perhaps Lévy's work on the Dirac delta in 1928 may have sown the seed of Schwartz's distributions.)
How thematic that it was a random event (the illness of Humbert) which (via Carvallo's invitation to give the three EP lectures of 1919) led Lévy to probability.
In sum: the giant of probability theory is clearly A. N. Kolmogorov (1903-87). Below only Kolmogorov, among the founding fathers the next two names are Alexander Yakovlevich Khinchin (Khintchine) (1894-1959) and Paul Lévy. If one must rank these two great probabilists, toss a coin. Over all time, the next name is Kiyosi Itô (1915-2003). Itô was both very much himself, and the "Lévy of the next generation". One can't rank these two at all: both names are on both sides of the coin.

NHB, 15.12.2011