PROBABILITY EVERYWHERE

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Probability and Mathematics

Mathematics is the common core of science. Science is what distinguishes between the modern world and the Middle Ages. Without mathematics there is no science to speak of, and without science we're back in the Middle Ages, burning witches.

The essence of mathematics is proof. If what one is doing focusses on proof, it is essentially Pure Mathematics. If not, it's probably Applied Mathematics. This is not said in criticism: the great physicist Paul Dirac was once accused of not having proved something. He replied "I am not interested in proof; I am interested in what Nature does". In passing: students tend to split into people who like proofs and people who like calculations. To any who don't much like proofs: saying "I love mathematics, it's proofs I can't stand" – though very human – is about as sensible, or as silly, as saying "I love humanity, it's people I can't stand".

To a probabilist, such as myself, mathematics tends to split into the deterministic part and the random part. The random part tends to split between Probability (where one is given the mechanism generating the randomness and studies its consequences), and Statistics (where one is given data, and studies what the data has to tell us about the (random) mechanism that generated it). Thus Probability is (or at least, I see it as) intermediate between Mathematics and Statistics.

One of the nice things about Probability is how versatile it is. One can be as pure as one likes, or as applied; one can be as statistical as one likes, or not; one can be as analytical, as algebraic, as geometric, as combinatorial, etc. Indeed, Probability is so many-sided and flexible a subject that it illustrates that to draw a distinction between pure and applied mathematics is a false dichotomy. It is less a question of content than of emphasis.

One surprise is how probability comes into everything, even things that on the face of it it has nothing to do with. For example, one could not think of anything more God-given, or deterministic, than the set of natural numbers, whose very simplicity of definition has led to them being studied in fantastic detail – Number Theory. In Statistics, one encounters the normal distribution – the signature of randomness in a continuous setting (it arises when what we see is the superposition of a very large number of causes, each with an individually negligible effect)[1]. In mathematical language, this is the Central Limit Theorem (CLT); in physical language, this is the Law of Errors ("errors are normally distributed about the mean"). In 1939, Erdös and Kac were studying how many prime factors a natural number n has. Counting with or without multiplicity, they found that, in a sense, the resulting counts $\omega(n), \Omega(n)$ behaved like normally distributed random variables with mean and variance $\log \log n$. Kac memorably summarised this by saying "Primes play a game of chance". Equally, study of the natural numbers will reveal that the primes seem to be very irregularly distributed. This was memorably summarised by my contemporary Bob (R. C.) Vaughan (then of Imperial College): "It's obvious that the primes are randomly distributed – it's just that we don't know what that means yet". Or as my wife Cecilie summarised both of these: "Primes play a game of chance – we just don't know the rules yet".

Incidentally, it then became obvious that the German number theorist Edmund Landau had done a variant on this much earlier, in 1900. What he found was not the normal, but the Poisson distribution – the signature of randomness in discrete situations. But, great mathematician though he was, Landau didn't know any probability.[2]

I mention here in passing an application of probabilistic ideas to analysis that turned me into a probabilist. It concerns not the CLT as above, but something even more basic – the Law of Large Numbers (LLN). This result (really a whole family of theorems – strong LLN, weak LLN, etc.) is the mathematical counterpart of what the man or woman in the street calls the "Law of Averages". Some time as an undergraduate (Oxford, c. 1965), I saw in lectures (by J. S. (Jack) de Wet, as I recall) the proof of Weierstrass'

[1] There is a picture of Gauss, the greatest mathematician who ever lived, and of the (standard) normal – Gaussian – density curve, on the pre-euro 10 DM banknote.

[2] Nor did Hardy – if he had, he would have discovered the Law of the Iterated Logarithm, as Erdös observed.

approximation theorem by Bernstein polynomials. If $f \in C[0,1]$, and

$$f_n(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n),$$

then

$$f_n(x) \to f(x) \qquad \text{uniformly in } x \in [0,1].$$

The proof reduces (apart from f being bounded and uniformly continuous) to the weak law of large numbers. What struck me immediately, and dramatically, was that if one knows some probability, one can do (at least some kinds of) analysis better than analysts can. I resolved to make the interface between analysis and probability my natural academic habitat.

Probability everywhere: some examples

Apart from number theory, there are many other areas of pure mathematics where probability appears, 'unexpectedly'. We give a few examples.

1. *Brownian motion on manifolds.* Brownian motion is the archetypal stochastic process (stochastic: random; process: unfolding with time). Riemannian manifolds are important in geometry (differential geometry), and general relativity. There is a whole subject on the interplay between the geometry and topology of a manifold, and the probabilistic properties of Brownian motion on it.
2. *Random walks on groups.* Again (and this is partly a discrete analogue of the above), there is a great deal known on the interplay between the algebraic properties of groups, and the probabilistic properties of random walks on them.
3. *Potential theory.* This subject originated in physics (Green's *Essay on magnetism and electricity* of 1828), but has since permeated mathematics, particularly complex analysis and probability theory.
4. *The probabilistic method.* This originated with Erdös and his many collaborators; the standard work on this is by Alon and Spencer (3rd ed., 2008). The idea is to show that certain kinds of behaviour, which it may be very hard to exhibit explicitly, not only exist, but are generic.
5. *Geometry of Banach spaces.* One can classify the geometry of a Banach space B according as to which probability limit theorems hold on it. For example: the Radon-Nikodym Theorem and the Martingale Convergence Theorem are the same theorem, in that (they hold in Euclidean and Hilbert spaces and) one holds in a Banach space B iff the other does. Equally, for

$p \in [1, 2)$, the Marcinkiewicz-Zygmund Law of Large Numbers holds in B iff B has type p.

6. *Randomized algorithms.* One knows from public-key crytography that knowledge of extremely large prime numbers is crucially important in such areas as computer security (it provides a 'trap-door function', making something easy if one knows the key fact but inaccessibly hard if not). It turns out that randomizing algorithms in this area can be very useful.

7. *Operator theory and stationary processes.* The nicest spaces in Functional Analysis are Hilbert spaces. The best developed part of Hilbert-space theory is Operator Theory – the extension of Linear Algebra from finite to infinite dimensions. The fundamental map in stationary stochastic processes (in discrete time, say) is moving forward one time-step. This can be identified with the *shift operator* in a Hilbert-space setting, to very good effect.[3]

Here are a few applications of probability in statistics.

8. *MCMC* (Markov chain Monte Carlo). The probabilistic subject of limit theorems for Markov chains has proved enormously useful in statistics, where it provides one with efficient means of simulating.

9. *Empiricals.* The theory of empiricals (distributions and processes) arises in probability theory, where it leads to theoretical subleties involving outer measure rather than measure, etc. All this turns out to be extremely useful in non-parametric statistics.

10. *Epidemiology.* The mathematics of the spread of infectious diseases is extremely important, and very interesting (AIDS/HIV, foot and mouth, SARS, bird flu, ...) – in humans, in animals (swine flu, etc.), plants (Dutch elm disease, ash die-back, etc.).

11. *Bioinformatics.* The Human Genome Project has had great impact, and holds out great prospects for medical advance. It involves a great deal of probabilistic modelling, statistical analysis, heavy computation, liaison with medical people, etc.

A further application, from a variant of Fourier analysis to both probability and statistics is

12. *Wavelets.* This is the machinery needed for efficient compression of many kinds of data set. For instance, wavelets were used to digitise the FBI's finger-print bank (without which, with over 2 million in prison in the USA, the US criminal justice system would have collapsed long ago).

There is lots of mathematics to be interested in (too much for comfort,

[3]For background, see e.g. my two surveys in *Probability Surveys* 9 (2012).

if anything), but a good mathematician should be interested in other things too, including science. I want to say a word or two here about physics. The quantum age goes back to 1900 (Planck). Newtonian mechanics is fine for macroscopic objects (and is deterministic); at the subatomic level, one needs quantum mechanics. It has been known since Born in 1926 that quantum mechanics is probabilistic (Einstein famously refused to believe that "God plays dice with the universe" – but He does, and Einstein's stubborn refusal to accept this cost him decades of lost effort). To study the world one needs to model it, and realistic models must be probabilistic.

Rao and the sex ratio

The greatest living statistician is Professor C. R. Rao, and I want to tell you my favourite Rao story. Rao was speaking at a conference in the University of Sheffield, twenty-odd years ago. I knew he was a great man; I hadn't realised he is a showman at heart. He began by thanking the two conference organisers, both present, and asking them to assist him in one small task. The room was packed, and divided down the middle; each organiser was asked to count hands, one on each side. Rao asked everyone present who had a brother to raise their hand. A forest of hands went up; these were carefully counted, and the totals given to Rao, who wrote them on the board and added them. He then asked everyone present who had a sister to raise their hand. Many hands went up – but it was immediately obvious, to everyone, that *far fewer* hands went up. A collective gasp of astonishment went up, and it was obvious that everyone present was flabbergasted, *except Rao*. He then proceeded to explain this. One needs to know two things. The first we all know. Academic subjects show a strong gender bias. In maths, the sex ratio may be around 50:50 at undergraduate level, but at postgrad level it's maybe 60:40, postdoc maybe 70:30, lecturer maybe 80:20, reader maybe 90:10, and at professorial level it is actually around 95:5. The second relevant fact I didn't know (despite having fathered two children then and three now). While sperm production is to a first approximation 50:50 between male- and female-producing sperm, so 'which sperm?' is a coin-toss, at couple level things are asymmetric. Some couples are predominantly boy-producing; these are balanced by some couples being predominantly girl-producing. On being told this, I immediately realised that I had seen many examples, as I suspect you have too. The rest you can see coming. This was a distinguished mathematical audience; so, by above, a predominantly male audience. So, the parents were sampled, not from the population of all parents, but predominantly from the population of *male-producing parents*. This is an example of an insidious statistical danger known as *selection bias*.

There is a tail-piece to this. For years, I simply accepted the second fact, but had the wit to ask a doctor friend over a drink once what the reason was. He laughed, and said, 'It's sex, Nick'. We are both fathers (our wives, who are best friends, met in the maternity hospital), so I told him that I was aware that we mammals reproduced sexually, and asked for the mechanism. I should have known: I was aware that vaginal ph varied during the menstrual cycle, and that one of acid or alkali favoured one of boys or girls. In the tone of someone explaining the facts of life to a grown man, he went on to explain that a woman's libido also varies during the cycle, but in different ways for different women .... Obvious enough, really – but only with hindsight – like so many things in science.

Two professions.

I want to say a word about two professions (outside the educational sphere, where I hope a number of you will choose to go – at school or university level!) where probability plays a vital role.

The first is the *actuarial* profession. This deals with the quantitative side of the insurance industry – which whether life or non-life (i.e., whether claims are triggered by death or by accident) is probabilistic. The basic model of the random point-process of claims is by a Poisson process, and of the random process of amount claimed to date is by a compound Poisson process. The classic ruin problem deals with the probability that the company's cash reserve (the premium income – linear and deterministic – less the amount claimed) goes negative (ruin), as a function of the initial capital.

The second is *mathematical finance*. You have probably heard of the *Black-Scholes formula* (of 1973). The reason for its importance is not so much that the formula is correct (Fischer Black himself famously wrote a paper called *The holes in Black-Scholes*), but that it gave for the first time an answer to an important question: what is the value of an option? (An option is a financial derivative, giving one the right but not the obligation to buy or sell a risky stock at some time in the future, at some specified price). The market in derivatives is now much bigger than the market in the underlying stock (a signal that the financial system was less stable than it looked, at least back in 2007). There are also options on interest rates, foreign exchange etc. (the bond markets are nowadays even more important than the stock markets). The people who do the mathematics here are variously known as quantitative analysts (quants), financial engineers and rocket scientists. The phenomenal growth in the area has had a powerful lure on large numbers of young mathematicians (and is one reason why I teach a lot of the mathematics I do to large audiences, intending to go to work in the City, rather than to small ones, studying for interest or academic reasons, some decades ago.

There are lots of good technical problems to work on these days in mathematical finance. But stand well back, and look at things from the perspective of the public generally, or society at large. What is really crying out for attention is not technical mathematics, however interesting, but the horrible damage inflicted on the world economy by the financial crises of 2007 on. Regulation is clearly vital here; so are macro-prudential issues generally. Underlying causes (stupidity and greed apart) included the grotesque geo-economic imbalances, which grew unsustainable in our increasingly interconnected world. All this is extremely important; everything important enough becomes political (Couve de Murville); politics is not an exact science (Bismarck). Incidentally, I and a distinguished colleague of mine in mathematical finance entertained a guest to lunch soon after the crisis began. He asked my colleague how he felt. He replied: "Rather foolish, actually – I feel as if I've just spent years studying an elephant's foot in detail, without noticing that the elephant was about to fall on top of me". Quite.

Who does what where?

A young mathematician casting around for a PhD area these days can (and should!) ransack the websites of the mathematics departments of a range of universities. Time spent on reconnaissance is never wasted (a military maxim I learned from my soldier grandfather, and it stuck). In some institutions, one needs to look in more than one place. I recommend a good browse in at least the following:
Oxford: Mathematical Institute; Statistics Department; OCIAM; Man Inst.
Cambridge: Statistical Laboratory
Imperial College: Mathematics Department (Pure Math. Section; Statistics Section; Math. Finance Section); Business School
Warwick: Maths Department; Statistics Department; Business School
LSE: Maths Department; Statistics Department
King's College London: Maths Dept
UCL: Maths Dept; Statistics Dept
University of Bath: School of Math. Sciences (Statistics & Probability)

Universities of Bristol, Sheffield, Leeds, Manchester, Edinburgh, Liverpool
Heriot-Watt University: Dept. of Actuarial Mathematics and Statistics
City University: Cass Business School

Traditionally, actuarial mathematics has been strong in City (London) and Heriot-Watt (Edinburgh). Financial mathematics is more recent (this millennium); the main groups are those with MSc programmes. If you want to explore this, you should also look at financial matters more generally, economics, etc., but we will not pursue this further here for lack of time.

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