TIME: A MATHEMATICIAN'S VIEW

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1. A tale from the silent cinema

When I was a boy, the silent cinema was still remembered, and once a year my school was visited by the local police (the then West Riding Constabulary) for road safety instruction. To sweeten the pill, we were afterwards shown, as a treat, selected extracts from silent films. Two stick in my mind. The first was of indians pursuing a covered waggon, with the US Cavalry arriving just in the nick of time (the scientific importance here: the stroboscopic effect of the spokes of the wheels, which seemed to rotate slowly backwards, a result of the 16 frames per second of the film). The second showed a mustachioed villain, who had tied a beautiful heroine across the railroad tracks somewhere out in the Rockies. Meanwhile, a steam train was approaching from round a great bend. There was a sustained montage, cross-cutting between the sadistic villain, curling his waxed mustaches, the beautiful heroine hollerin' and screamin', and the crew of the train, shovelling in coal etc. Suddenly the driver sees the heroine, and frantically pulls on the brake. There is a further montage, cross-cutting between the crew hauling on the brake and the heroine. The dramatic final shot is a view of the train head-on, as it approaches the heroine – and stops, just in the nick of time.

How did they do it?

I ransacked cyberspace for the name of the film (so as to look for it), without success.

Consider: error one way would have killed the beautiful heroine, leading to disgrace for the studio and punitive damages from her family, probably bankruptcy. Error the other way would have risked seeming anti-climactic after all that elaborate build-up.

Answer: the filmed the train starting from up against the heroine, reversing away from her – and showed this in reverse.

How could one tell?

Answer: *the steam was going INTO the funnel.*

One does not see this in real life. Why not?

Answer: the Second Law of Thermodynamics: *entropy increases.* Entropy is a measure of disorder: Nature evolves towards increasing disorder.

The term, and the insight, is due to Rudolf Clausius (1922-1888), in 1865, who formulated the Fist Law of Thermodynamics (Law of Conservation of Energy) and Second Law of Thermodynamics (entropy increases – things become more disordered), in the most famous two-sentence passage in the history of science:
1. Die Energie der Welt ist konstant (The energy of the world [the universe] is constant).
2. Die Entropie der Welt strebt einem Maximum zu (The entropy of the world [the universe] strives towards a maximum).

A tale from 1942 propaganda in WWII

The theme is the same – the *arrow of time* (see below).

As background, one has to know something about the Lambeth Walk, a popular dance, and song, turned into a 1937 musical (London revivals 1985-93, 2006-07):

Any time you're Lambeth way,
Any evening, any day,
You'll find us all,
Doing the Lambeth Walk – Oi!.

One can find the following clip (2:15, Charles A. Ridley) by Googling "Lambeth Walk – Nazi style". It illustrates the unsuspected popularity of the Lambeth Walk in Nazi Germany – as well as the arrow of time.

2. The Arrow of Time

The phrase *arrow of time* seems to be due to the British astrophysicist Sir Arthur Eddington (1882-1944) in 1927 (Eddington provided experimental confirmation of Einstein's General Theory of Relativity in a famous observation of a solar eclipse in 1919).

Eddington distinguishes several arrows of time:
the thermodynamic arrow – the direction in which entropy increases (by the Second Law of Thermodynamics);
the cosmological arrow – the direction in which the Universe expands (according to the Hubble red shift: the American astronomer Edwin Hubble (1889-1953) in 1929);
the psychological/perceptual arrow – we remember the past, but not the future.
There are in addition other arrows of time, in particle physics and quantum mechanics. Indeed, there is thought to be a quantum source of time (§5).

A good source for this material is one of the books of the (last) century,

Stephen W. Hawking, *A brief history of time: From the big bang to black holes*, Guild, 1988,
Ch. 9: The arrow of time.

There are also connections with evolution. See e.g.
Harold F. Blum, *Time's arrow and evolution*, 3rd ed., PUP, 1982 [1st ed. 1951, 2nd ed. 1955].

The arrow of time can, of course, be widely seen in literature. One famous passage suffices:

The Moving Finger writes; and, having writ,
Moves on: nor all thy Piety nor Wit
Shall lure it back to cancel half a Line,
Nor all thy Tears wash out a Word of it.

– Omar Khayyám (translation by Edward Fitzgerald).

There are also obvious connections with mortality. For example: the New Zealand expert in Operations Research Professor Peter Whittle FRS (1927 –) has written two autobiographical pieces. Of the first, in 1986, *In the late afternoon* (*The craft of probabilistic modelling: A collection of personal accounts*, ed. J. Gani, Springer, 186-195), I wrote "This haunting hint of brief mortality, conveyed in a mere four words, gives a good impression of the power and economy of Whittle's prose style" (the second, *Almost home*, in his Festschrift of 1994, hints not at mortality but at how a New Zealander feels in the UK).

Another literary illustration is the novel *Time's arrow* (1991) by Martin Amis (1949 –).

3. Reversibility

A stochastic process (stochastic: random; process: evolving with time) is *reversible* if the probabilistic structure is the same going forwards in time as going backwards in time – i.e., one cannot tell which way the arrow of time points.

For simplicity, restrict attention to Markov processes (roughly: for predicting the future, one need know only the present; the past is then irrelevant).

The laws of physics – specifically, Newton's Laws of Motion, applied to particles, say (*Principia*, 1687) are reversible: mechanics is the same forwards and backwards in time. Thus at *microscopic* level, we see reversibility.

At *macroscopic* level, we see irreversibility (Second Law of Thermodynamics): steam comes out of funnels, and doesn't go in; dropped china that breaks stays broken, and does not lift itself up and reassemble, etc.

One of the founding fathers of Statistical Mechanics, Ludwig Boltzmann (1844-1906), gave in his H-theorem of 1872 an explanation of the Second Law of Thermodynamics in terms of molecular motion (equivalence of preservation of equilibrium and detailed balance). The equivalence of reversibility and detailed balance is due to A. N. Kolmogorov (1903-87) in 1936. For background, see e.g.

R. H. Fowler, *Statistical mechanics*, CUP, 1929/1936, §17.11: Detailed balancing; §17.3: Boltzmann's H-theorem for a simple gas;

F. P. Kelly, *Reversibility and stochastic networks*, Wiley, 1979.

The *reversibility paradox* above, or Loschmidt's paradox, was posed by Josef Loschmidt (1821-1895) in 1876 (and by Lord Kelvin in 1874), and was followed by Boltzmann's entropy formula ($S = k \log W$ – on his gravestone) in 1877. In those days, the existence of atoms was questioned by some, particularly Ernst Mach (1838-1916) in 1897. It was a philosophical attempt to refute Mach that undermined Boltzmann's sanity and led to his suicide. See e.g.

David Lindley, *Boltzmann's atom*, The Free Press, 2001.

The first model specifically advanced to reconcile microscopic reversibility with macroscopic irreversibility was due to Paul Ehrenfest (1880-1933) and Tatjana Ehrenfest – the *Ehrenfest urn* (1907, Phys. Z., Encyclopaedia, 1911). In this, a number d of balls (or gas molecules, say) are divided between two urns (or the two halves of a room). At each stage, a ball is chosen at random and switched to the other urn. This model is reversible, and has a limit distribution (the binomial – as for coin-tossing). If all balls start in one urn (or initially one half of the room is a vacuum, separated from the other by a membrane, which is suddenly ruptured) – highly asymmetrical – this state will recur (indeed, infinitely often!), but its recurrence time is 2^d. If d is itself astronomically large – e.g., Avogadro's number, the scale factor (6.02×10^{23}) between the microscopic and macroscopic scales – this recurrence time is so enormous at to be effectively infinite.

Had the Ehrenfest model been available ten years earlier, Boltzmann might not have been driven to suicide.

Interestingly, a model similar to but harder than the Ehrenfest urn, namely the *Bernoulli-Laplace urn* (Daniel Bernoulli in 1770, Laplace in 1810-11, 1812), which also reconciled microscopic reversibility with macroscopic irreversibility, had long been available. But Bernoulli and Laplace were working long before Statistical Mechanics and Thermodynamics, so what with hindsight can be seen as the main thrust of their work was not noticed, and the work itself was forgotten, with tragic consequences for Boltzmann. See Martin Jacobsen, Laplace and the origin of the Ornstein-Uhllenback process, *Bernoulli* **2**.3 (1996), 271-286.

4. Discrete v. continuous time

Is time discrete or continuous?

Note that the two are often handled separately, and at length. See e.g. J. L. Doob, *Stochastic processes*, Wiley, 1953, V, VI (Markov processes), X, XI (Stationary processes);
N. H. Bingham and Rüdiger Kiesel, *Risk-neutral valuation: Pricing and hedging of financial derivatives*, Springer, 2004/1998, Ch. 3,4; Ch. 5,6.

Time is both (but see §5). Which one works with depends partly on mathematical context or convenience, partly on how one *measures* time – whether one has a watch with a sweep second hand (continuous – simple harmonic motion), or a digital watch.

One area in which the discrete v continuous time dichotomy is crucial is *Time Series* in Statistics. Here, one passes between the *time domain* – handling the time dependence in one's process $X = (X_t)$ explicitly, e.g. via the autocorrelation function – and the *frequency domain* – working via the Fourier transform, the ideal tool to detect hidden frequencies, etc. with the spectral measure – by the *Kolmogorov isomorphism*, $X_t \leftrightarrow e^{it\cdot} = (e^{it\theta})_\theta$. For mathematicians: if time is discrete, with time-set $\mathbb{Z}$, say, the spectral measure lives on the unit circle or torus $\mathbb{T}$; if time is continuous, the time-set is $\mathbb{R}$ and the spectral measure lives on $\mathbb{R}$ – both by Pontryagin duality (L. S. Pontryagin (1908-1988) in 1934: if the (locally compact abelian) group G is discrete/compact, its dual $\hat{G}$ is compact/discrete).

5. Quantum aspects

Max Planck (1858-1947) was led in 1900 to introduce the *quantum* concept by his work on black-body radiation. For background, see e.g.
Thomas S. Kuhn, *Black-body radiation and the quantum discontinuity, 1894-1912*, U. Chicago Press, 1987 [OUP 1978].

In 1899, Planck introduced what is now known as the *Planck scale* (the term is due to J. A. Wheeler in 1955). This is based on three fundamental physical constants:
the *gravitational constant* G, from Newton's Inverse Square Law of Gravity: $G = 6.67 \times 10^{-11} m^3/kg\ s^2$;

the *speed of light*, $c = 3 \times 10^8 m/s$;
Planck's constant h; $\hbar := h/2\pi = 1.05 \times 10^{-34} kgm^2/s$
(energy E and frequency ν are linked by $E = h\nu$, *Planck's law* or the *Planck-Einstein relation*).

We know from Einstein's General Theory of Relativity (Albert Einstein (1879-1955) in 1916) that space and time are linked: we should think in terms of space-time. From quantum theory, space is discrete; so time is discrete also; so too is energy, and (from the Einstein relation $E = mc^2$) so too is mass. The Planck scale comprises:
the *Planck length*, $\ell_P := \sqrt{\hbar G/c^3} = 1.616 \times 10^{-35}$ (about 10^{-20} times the size of a proton);
the *Planck time*, $t_P = 5.391 \times 10^{-44}$ s (time for light to travel a Planck length);
the *Planck mass*, $m_P := \sqrt{\hbar c/G} = 2.176 \times 10^{-8}$ kg.
the *Planck energy*, $E_P := c^2 m_P = 1.22 \times 10^{28}$ eV.
On the Planck scale, one cannot separate the effects of gravity from those of the other three fundamental forces of Nature (electromagnetism and the weak and strong nuclear forces). This is central to the emerging field of *quantum gravity*. Recall that the two great scientific advances of the 20th C. were quantum theory and relativity; quantum theory is needed for the very small (subatomic particles etc.); relativity is needed for the very big (cosmology etc.); each is clearly right rather than wrong – but the two are incompatible. Hence probing the nature of time is inseparable from – indeed, is much the same as – the search for a Grand Unified Theory of the four fundamental forces of Nature. We must refer elsewhere for more here; see e.g.
Ronald J. Adler, Six easy roads to the Planck scale, *Amer. J. Phys.* **78** (2010), 925-932,
Lee Smolin, *Three roads to quantum gravity*, Weidenfeld & Nicholson, 2000,
C. J. Isham, Prima-facie questions in quantum gravity, arXiv:gr-qc/9310031v1 22 Oct 1993.
We close by noting that the last author cited, one of the prime movers in this fascinating field, is one of our own, Chris (C. J.) Isham, Physics Department, Imperial College.

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