TOPOLOGICAL REGULAR VARIATION

N. H. BINGHAM

Maths Dept., Imperial College

Liverpool University, Wednesday 29 February 2012

Joint work with A. J. OSTASZEWSKI, Maths Dept., LSE

17 "BOst" papers [1 2008, 6 2009, 8 2010, 2 2011, all on MathSciNet],
plus 9 "Ost" papers [1 2010, 1 2011, both on MathSciNet, + 7 preprints, one with Harry I. Miller – pdfs on his LSE homepage],
plus a book, "BOstaszewski", Cambridge Tracts in Math., being written.
For 'ordinary' – 'measurable' – regular variation, see
BGT: N. H. BINGHAM, C. M. GOLDIE and J. L. TEUGELS, *Regular variation*, CUP, 1989 (1st ed. 1987).

§1. Why regular variation?

Limit theorems in probability theory:
BGT, Ch. 8. Recall two basic results:
Weak Law of Large Numbers (WLLN): for X_n iid, distribution function F mean μ, $S_n := \sum_1^n X_k$,

$$S_n/n \to \mu \quad (n \to \infty) \quad \text{in prob.},$$

Central Limit Theorem (CLT): if X_n are iid with mean μ and variance σ^2,
$\Phi(x) := (1/\sqrt{2\pi}) \int_{-\infty}^x \exp\{-\frac{1}{2}y^2\}dy$,

$$(S_n - n\mu)/\sqrt{n}\sigma \to \Phi = N(0,1) \quad (n \to \infty) \quad \text{in law.}$$

How far can we generalize, beyond existence of the mean and variance? Both answers involve regular variation:
WLLN: S_n/a_n converges to a (non-zero) constant in prob. iff the *truncated mean* $\int_{-x}^x y dF(y)$ is slowly varying (sv);
CLT: $(S_n - a_n)/b_n$ converges in law to a (non-deg.) limit iff the *truncated variance* $\int_{-x}^x y^2 dF(y)$ is slowly varying.

Here $f : (0, \infty) \to (0, \infty)$ is *regularly varying* if

$$f(\lambda x)/f(\lambda) \to g(\lambda) \quad (x \to \infty) \quad \forall \lambda > 0;$$

for f measurable or with the Baire property $g(\lambda) \equiv \lambda^{\rho}$ for some ρ, the *index* of f, $f \in R_{\rho}$. Then $\ell \in R_0$ is *slowly varying*: $\ell(\lambda x)\ell(x) \to 1$.
Tauberian theory: BGT, Ch. 4,
J. Korevaar, *Tauberian theory: A century of developments*, Grundl. 329, Springer, 2004, IV: Karamata's heritage: regular variation.

$$\int_0^{\infty} e^{-sx} d(x^{\rho}) = \rho \int_0^{\infty} e^{-sx} x^{\rho-1} dx$$

$$= \rho \Gamma(\rho)/s^{\rho} = \Gamma(1+\rho)/s^{\rho} \quad (\rho > 0) :$$

the Laplace-Stieltjes transform of a power x^{ρ} is $\Gamma(1+\rho)/s^{\rho}$. The Hardy-Littlewood-Karamata theorem (H&L 1914, Karamata 1931; BGT §1.7) extends this from powers to regularly varying functions, both ways (for more general transforms, see BGT Ch. 4).
Also: *Analytic Number Theory*, BGT Ch. 6; *Complex Analysis*, BGT Ch. 7.

§2. Extreme-value theory (EVT)

See BGT §8.13, and the work of the 'three flying Dutchmen', Wim Vervaat (1942-94), Guus (A. A.) Balkema and Laurens de Haan:

H. Maassen & F. W. Steutel, Remembering Wim Vervaat, *Stat. Neerlandica* 50.1 (1996), 225-230.

EVT is crucially relevant to such practical problems as flooding in the Netherlands (see below). The original theory is univariate. Multivariate extensions are also needed (sea levels at a number of different points), and stochastic-process versions (sea levels along the entire dyke defence system). There have been recent book-length treatments:

Guus Balkema & Paul Embrechts, *High-risk scenarios and extremes: A geometric approach*, Zürich Lectures in Advanced Math., European Math. Soc., 2007,

L. de Haan & A. Ferreira, *Extreme-value theory: An introduction*, Springer, 2006.

From my contribution to the Jef Teugels Festschrift [Regular variation and probability: The early years. *J. Computational and Applied Mathematics* **200** (2007), 357-363, MR2276837 (2008g:26004, C. M. Goldie)]:

"Meanwhile, mathematics was overtaken by reality. On the night of 31 January to 1 February 1953, a storm surge in the North Sea caused extensive flooding and many deaths. In the UK, 307 were killed; in the low-lying Netherlands, 1,783 people were killed (over 1,800 on some counts). The author, then a schoolboy of seven, remembers the public shock at the time very well. The Netherlands Government immediately gave top priority to understanding the causes of such tragedies with a view to preventing them if possible. Since it is the maximum sea level which is the danger, EVT is immediately relevant, and thus EVT became a Netherlands scientific priority" (Balkema, de Haan, Vervaat).

§3. Regular variation, BGT

Karamata (1930, 1931): continuous functions
Korevaar, van Aardenne-Ehrenfest, de Bruijn (1949): measurable functions
Matuszewska (1965): Baire functions (property of)
BGT, 1987/89 (historical aspects: NHB, Jef Teugels Retirement Meeting, 2007).
Core theory rests on:
Steinhaus' Theorem (1920) (Baire: Piccard (1939); BGT Th. 1.1.1). If A is measurable and non-null [Baire and non-meagre], its difference set contains an interval [around 0].
Ostrowski's Theorem (1929) (Baire: Mehdi (1964); BGT Th. 1.1.8). If k is an additive function, bounded below [or above] on a non-null mble [non-meagre Baire] set, $k(x)$ is of the form cx: and so continuous – *automatic continuity* [k is 'very nice or very nasty' – *dichotomy*]:
[BOst, Dichotomy and infinite combinatorics;

the theorems of Steinhaus and Ostrowski, MPCPS 150 (2011), 1-22, MR2739070].

Uniform Convergence Theorem (UCT). If $\ell : (0,\infty) \to (0,\infty)$ is mble/Baire and

$$\ell(\lambda x)/\ell(x) \to 1 \quad (x \to \infty) \quad \forall \lambda > 0, \qquad (SV)$$

then the convergence is *uniform* on compact λ-sets in $(0,\infty)$ (*false* with no condition on ℓ).

Questions arising from BGT:

Foundational ('gap at the beginning': p.11, §1.2.5). What is the proper context (or minimal common generalization of mble and Baire)?

Contextual ('gap at the end': p.423-6, Appendix 1). What is the natural generalization of $(\lambda, x) \mapsto \lambda x : (0,\infty)^2 \to (0,\infty)$ [or $(u,v) \mapsto u+v : (-\infty,\infty)^2 \to (-\infty,\infty)$]?

Hard proofs. (i) To reduce the number of hard proofs to zero.

(ii) Seneta-Heiberg Theorem [Seneta (1973), Heiberg (1974)]: Th. 1.4.3, p.18-19 in the Karamata case, proved as Th. 3.2.5, p.141-3 in the de Haan case]. Why is this so hard? Simplify!

§4. **Bitopology**.

A key new insight of our approach is that *it is the Baire case that is primary, not the mble case*. We handle the two together as ‘both Baire’, using *two* topologies: the Euclidean topology $\mathcal{E}$ for the Baire case, and the *density* topology d (below) for the mble case. Recall that for A mble, a is a *density point* of A if

$$|A \cap (a-\delta, a+\delta)|/(2\delta) \quad 1 \quad (\delta \downarrow 0).$$

By the Lebesgue Density Theorem, almost all points of such an A are density points. Call a (mble) set U d-*open* if all its points are density points. This defines a topology, the *density topology* or d-topology (not metric! – ‘d for density’). We quote:

(i) d is finer than $\mathcal{E}$ (d is a *fine topology*).

(ii) A is d-Baire iff it is (Lebesgue) mble.

(iii) A is d-meagre iff it is null. Hence

(iv) The reals form a Baire space under d (Baire’s Category Theorem holds, for both d and $\mathcal{E}$).

(v) A function is d-continuous iff it is (Denjoy) approximately continuous.
For details, see BOst, Beyond Lebesgue and Baire II: bitopology and measure-category duality, Colloq. Math. 121 (2010), 225-238, MR2738939.
Recall that completeness is not a topological concept (it is not preserved under homeomorphism) – but we need it to do analysis properly (think of ordinary calculus and the taking of limits). We follow
Ost, Shift-compactness in almost analytic submetrizable Baire groups and spaces, preprint.
A topology is *metrizable* if it is homeomorphic to a metric space. A refinement of a metrizable topology is *submetrizable* (not supermetrizable – be economical!). Topologies that are analytically generated are useful, and for topological groups those which are *almost analytic* – have a (norm) non-meagre analytic subset. This is equivalent to *almost completeness* – the topological version of completeness.

§5. Measure-category duality

Recall that there is a classic textbook treatment of this:

J. C. Oxtoby, *Measure and category: A survey of the analogies between topological and measure spaces*, 2nd ed., GTM 2, Springer 1980 [1st ed. 1971].

Recall also that:

(i) Countability is built into measure theory [via σ-additivity], Baire category theory [meagre := countable union of nowhere dense sets], metric spaces [via metrization theorems], and generalizations – but not into General Topology.

(ii) Regular variation is a *continuous-variable* theory – but most of the proofs in BGT of UCT proceed via contradiction and *sequences* witnessing to this contradiction.

(iii) There is a theory of *sequential regular variation*. BGT §1.9, Th. 1.9.1 (Kendall (1968), Kingman (1964); Croft (1957)):

If $\limsup c_n = \infty$, $\limsup(c_{n+1} - c_n) = 0$:

(i) If G is open and unbounded above, then for any open interval I there exists $x \in I$ with $c_n + x \in G$ i.o.
(ii) If f is continuous and $\lim_n f(x + c_n)$ exists for all x in some open interval I, then $\lim_x f(x)$ exists.
This involves *Baire category*, and 'bridges the gap' between discrete and continuous limits.
The topological subtleties here have been probed in depth recently by A. J. Ostaszewski:
Analytically heavy spaces: analytic Cantor and analytic Baire theorems. Topology and its Applications 158 (2011), 253-275,
Analytic Baire spaces [preprint].
The 'i.o.' above involves *infinite combinatorics* (Erdös and school).

6. Analytic sets

For background, see e.g.

C. A. Rogers et al., *Analytic sets*, Acad. Press, 1980 [Proceedings, LMS Conference, UCL, July 1978];

A. S. Kechris, *Descriptive set theory*, GTM 156, Springer, 1995, III.

The *analytic sets* – the 'nice' sets, of descriptive set theory – are the continuous images of Polish spaces. Analytic sets have the Baire property, and are (universally) mble. So we can combine the mble and Baire cases by specializing to analytic sets.

Think of the [mble] null and the [Baire] meagre sets as *negligibles*, forming a class $\mathcal{N}$. These form an ideal, indeed a σ-ideal [by $\Sigma 0 = 0$, or the Baire Category Theorem]. For analytic sets + σ-ideals of negligibles, see e.g.

BOst, Automatic continuity by analytic thinning, PAMS 138 (2010), 907-919,

and the two Ost papers cited earlier.

For a sequence $\sigma = (\sigma_1, \ldots, \sigma_n, \ldots)$ of natural numbers (note that there are *uncountably many* such σ – they are a model for the irrationals, under continued fractions), write $(\sigma|n) := (\sigma_1, \ldots, \sigma_n)$. For a class $\mathcal{S}$ of sets, the *Souslin operation* $\mathbf{S}$ yields the class

$$\mathbf{S}(\mathcal{S}) := \cup_\sigma \cap_n S(\sigma|n), \quad S(\sigma|n) \in \mathcal{S}.$$

The Souslin operation $\mathbf{S}$ is idempotent, and preserves the class of analytic sets (Lusin & Sierpinski 1918, Sierpinski, 1933). It also preserves the classes of Baire sets (Nikodym, 1925) and mble sets (Szpilrajn-Marczewski, 1929, 1933). It is crucial to handling analytic sets (see e.g. the books by Rogers and Kechris above). It is also the key tool by which we are able to handle uncountability in the theory – e.g., in passing between continuous and sequential limits.

7. Groups and actions

Bajsanski & Karamata (1968-9) began the study of regular variation on topological groups. But one must be careful: the reals do *not* form a topological group under the density topology d – although translation [the two-argument operation of addition specialised to one argument by fixing the other] *is* d-continuous. Topologists study such situations – semi-topological groups, paratopological groups, etc.

Group-norms are like vector-space norms, except that the scalars are restricted (to the units ± 1 in the abelian case, and the powers ± 1 generally). One can extend the theory of regular variation from the classical setting on the line [BGT etc.] to metrizable topological groups [note the countability implicit here]. Normed groups (V. L. Klee, 1952) are groups carrying a right-invariant metric (by the Birkhoff-Kakutani metrization theorem, a first-countable Hausdorff group has a right-invariant metric).

Normed groups show a *dichotomy*: they are either *topological groups* or *pathological groups*: BOst, Normed versus topological groups: dichotomy and duality. *Dissertationes Mathematicae* 472 (2010), 138p.
One can also work with group actions, regarding λ as acting on the group, $\lambda : x \mapsto \lambda x$. One can use the viewpoint of topological dynamics. See e.g.
Ost, Regular variation, topological dynamics and the Uniform Boundedness Theorem, Topology Proceedings 36 (2010), 305-336, MR2643693.

§8. Infinite combinatorics

Perhaps the most famous result here is van der Waerden's theorem (1927): in any finite colouring of the natural numbers, at least one colour contains arbitrarily long arithmetic progressions. This is one of Khinchin's three pearls of number theory:

A. Ya. Khinchin, *Three pearls of number theory*, Dover 1998 [Russian, 1947, 1948].

Infinite combinatorics has grown spectacularly, largely under the influence of Erdös and his school. See e.g.

T. Tao & V. N. Vu: *Additive combinatorics*, CUP, 2006.

The following result – the Kestelman-Borwein-Ditor Theorem (KBD) has proved very useful to us, e.g. in producing a short new proof (the 9th) of the UCT:

THEOREM (KBD: Kestelman 1947, Borwein & Ditor 1978, Trautner 1987; cf. the footnote on p.10 of 'BGT2', 1989). If $z_n \to 0$, T is measurable and non-null/Baire and non-meagre, then for all $t \in T$ off a null/meagre set, there is an infinite set M_t such that

$$\{t + z_m : m \in M_t\} \subset T.$$

There have been several developments of this in BOst papers – e.g., Category Embedding Theorem (CET). The best version so far is the 'bitopological shift-compactness theorem': Harry I. Miller & A. J. Ostaszewski, Group action and shift compactness, preprint.
See also §6 in
BOst: Kingman, category and combintorics, Ch. 6, p.135-168 in
N. H. Bingham & C. M. Goldie (ed.): *Probability and mathematical genetics.* Sir John Kingman Festschrift, LMS LNS 378, CUP 2010.

§9. Logical assumptions

Recall that ordinary mathematics uses Zermelo-Fraenkel set theory (ZF). We augment this by the Axiom of Choice (AC) when we wish/need to (ZFC). Recall also that, although 'most sets' (in the reals, say) are non-measurable, we need AC to be able to exhibit explicitly a non-measurable set (e.g., Vitali's example).

Recall also that in BGT Ch. 1 – Karamata theory – we worked with mble/Baire functions, and stayed within this class, essentially because sequential limits of mble/Baire functions are mble/Baire.

In BGT Ch. 2, Further Karamata theory: $R \subset ER \subset OR$ – we did not assume existence of limits, but allowed lim sup and lim inf, thereby introducing two pairs of indices, the Karamata (ER) and Matuszewska (OR) indices. We write

$$f^*(\lambda) := \limsup_{x\to\infty} f(\lambda x)/f(x),$$

and similarly for f_*. The theory is harder, as here measurability or the Baire property may

be lost. It rests on the version of the UCT due to Delange (1954), Csiszár & Erdös (1964) [BGT Th. 2.0.1].
One can extend our topological approach to this setting also. For details, see
BOst, Regular variation without limits, JMAA 370 (2010), 322-338, MR2651656.
It turns out that the Delange result *disaggregates*: one has to take it apart. Parts of it need only ZFC. Parts of it need stronger set-theoretic assumptions (Gödel's Axiom of Constructibility, the Axiom of Projective Determinacy, etc.) We probe the degree of *degradation* in going from f to f^*, from the point of view of descriptive set theory (First, Second and Third Character Theorems).

Historical note: The BOst collaboration was conceived at UCL in July 1978 (Conference on Analytic Sets) – before writing BGT began in March 1981.