Epsilon constants.

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Epsilon constants are funny things. They are attached to (finite dimensional, complex) representations of Galois groups of local and global fields, and also to lots of other things (elliptic curves, for example). Perhaps to anything that has an L-function? You need to choose a measure in the local case, but everything cancels out in the global case if you do it right. Deligne wrote a paper "Les constantes des equations fonctionelles des functions L" in Antwerp II and I just read some of it. I'll tell you about local epsilon factors for non-arch local fields; these seem to have no natural local definition.

1 Relations coming from Brauer's theorem.

If G is a finite group, then let $R(G)$ be its Grothendieck group, that is, the free abelian group with basis the finite dimensional complex representations of G, quotiented out by short exact sequences in the usual way.

Let $R^+(G)$ denote the free abelian group with basis the set of isomorphism classes of pairs (H,χ) where H is a subgroup of G and χ is a 1-dimensional representation of H. Define a group homomorphism $R^+(G) \to R(G)$ by sending (H,χ) to $\mathrm{Ind}_H^G(\chi)$. By Brauer's theorem this is a surjective group homomorphism. One can do better. Tensor product induces a multiplication on $R(G)$ making it a commutative ring (and even something Grothendieck called a λ-ring, although I don't know what that means). There is also a (totally messy but natural) multiplication on $R^+(G)$ (think of the proof that the tensor product of two induced representations is naturally a sum of induced representations) which makes the map $R^+(G) \to R(G)$ a surjective homomorphism of rings. The kernel is an ideal of $R^+(G)$. Of key importance for Langlands is working out generators for this kernel as an abelian group, because one way (Langlands' way) of proving that ϵ constants are well-defined in the non-arch local case involves (and is essentially equivalent to) checking that certain functions vanish on this kernel. Langlands wrote an infinitely long paper on ϵ constants which was never published and in it he proved some facts about generators of that kernel. Deligne does not use this approach at all—he uses the kernel, in a mild way, but not its generators. Deligne gives an exposition of some results of Langlands on this kernel anyway, in the last few sections of chapter 1 of his paper, although he does not need these results in his construction of local epsilon constants, which is a global construction. Deligne (following Langlands) writes down three kinds of elements in this kernel, all essentially explicit. They are called elements of type I, II and III. He proves (again following Langlands)

Lemma 1.13.1: if G is abelian then the kernel is generated as an abelian group by relations of type I.

Theorem 1.13: if G is nilpotent then the kernel is generated as an abelian group by the relations of type I and II.

Theorem 1.14: If G is solvable then the kernel is generated by relations of type I, II and III.

I think that Langlands went much further than this, analysing generators of the kernel for "des groupes convenables G" (arbitrary groups G, right?).

2 Finite dimensional complex representations.

2.1 Local fields.

If K is a local field (that is, $\mathbf{R}$ or $\mathbf{C}$ or a finite extension of $\mathbf{Q}_p$ or $k((t))$ for a finite field k), then fix a non-trivial continuous group homomorphism $\psi : K \to \mathbf{C}$ and a Haar measure d^*x on K^*.

2.1.1 The 1-dimensional case.

Given $\chi : K^* \to \mathbf{C}^*$ continuous, it has a local L factor $L(\chi) \in \mathbf{C} \cup \infty$, given by the special value of a Γ function if K is arch and a trivial formula if K is non-arch (it's 1 if χ is ramified and $(1-\chi(\varpi))^{-1}$ if not, where ϖ is a uniformiser). It also has a local ϵ factor $\epsilon(\chi, \psi, dx) \in \mathbf{C}^\times$, whose definition is perhaps more subtle than you expect. It's the ratio of two of Tate's local integrals, divided by the ratio of the two local L-functions. Note in particular that when K is a p-adic field then Tate's local integral is not equal to the local L-function in the ramified case, because the local L-function is just 1. More precisely, in the unramified case Tate's integral (with respect to a suitable test function) is just the Euler factor, as is the local L-function, so everything cancels. But in the ramified case the integral is related to Gauss sums and the local L-function is 1, so you get a Gauss sum for the local epsilon factor. (Recall that Tate's global theorem is that (after analytic continuation) the integral of a test function equals the integral of its Fourier transform, and when translated down into an equation about L-functions, the ratios of the local integrals versus the local L-functions is explained by the epsilon factor, which is visibly the product of the local epsilon factors).

The epsilon factor is an integral power of i if K is arch, and, as I said, essentially a Gauss sum if K is p-adic and χ is ramified. If χ is unramified then it's 1. The L-functions and epsilon factors are multiplicative in short exact sequences.

Note that this could be regarded as L and ϵ factors for 1-dimensional representations of Weil groups—but this invokes a hard theorem (isomorphism of the Weil group abelianised with K^*, where we fixed the Haar measure) so it's not surprising that things get harder in the general case, we can't really integrate over the full Weil group I guess.

2.1.2 The general case: finite-dimensional complex representations.

(Deligne 3.7) If K is local and ρ is a finite-dimensional continuous complex representation of $W(\overline{K}/K)$ then, in the arch case it's easy to extend the definition of L-functions, as every irreducible is induced from a character on another Weil group.

The ϵ factors are more subtle however. You don't have Tate's immense flexibility in choosing test functions and integrating over multiplicative measures in the non-arch case because you're dealing with the full Weil group. So even though there's a local L-function there's no obvious local definition of a local epsilon factor as a ratio of integrals divided by a ratio of these local L-functions.

One wants epsilon factors to behave well with respect to short exact sequences, change of multiplicative measure, induction (in degree 0), and to agree with Tate's definition in the 1-dimensional case. Clearly there is at most one such collection of ϵ factors, by Brauer. In the arch case you can just write them down and check they work. In the non-arch case the key problem is that you could try to write them down using Brauer, but then to prove that they work you have to check that the epsilon factors of anything of virtual dimension 0 in the kernel mentioned in the previous section, is 1. This is what Langlands did in his paper of infinite length.

2.2 Global fields.

2.2.1 The 1-dimensional case.

For a grossencharacter of a global field, the global epsilon factor is just the product of the local ones. As χ varies in one of Tate's families (that is, change χ to $\chi.||^s$) the ϵ factor is of the form

$A.e^{Bs}$.

2.2.2 The general case.

The point here is that things work quite well! There is a well-defined global epsilon factor. The L-functions are well-defined objects and have meromorphic continuation via Brauer, and you can define the global epsilon factor in the n-dimensional case as the ratio of two L-functions. Global L-functions behave well under induction and so on, and global epsilon factors have all the properties you want them to have. Note that their definition is via the L-function in some sense.

3 What Deligne did.

Deligne observed that basically if L/K was a finite extension of global fields, and local epsilon factors worked well for all but one completion, then they worked well for the other completion. This takes a bit of work before you can formulate this rigorously but it's not deep, just clever. The proof is basically trivial, coming from good behaviour of the global epsilon constants. Local epsilon factors "exist in the unramified case", so now the construction of local epsilon factors boils own to an easy argument constructing global fields from local ones, and then a clever twisting argument.

Tate's Corvallis article shows that if you choose the measure in the non-arch case such that the measure of the integers is 1, then there's a nice Galois-invariance property of the epsilons, by unicity!