Gouvea's thesis: a summary.

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Gouvea's thesis (Springer LNM 1304) is something that I used to try and struggle through when I was trying to understand Coleman theory. I just tried to re-read it with the benefit of hindsight and found it rather easy-going, so I thought I'd summarise it quickly before I forgot it all.

1 I.1–I.2.

Gouvea (Note to me: globally replace this with Gouvea with a hat on the e) follows Katz and hence gets interesting integral structures on overconvergent modular forms, but his development of the theory suffers from the same constraints (for example he cannot define an overconvergent form of weight κ for $\kappa : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ with image not in $\mathbf{Z}_p^\times$; note however that he does give a definition of overconvergent forms of weight κ for $\kappa : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$; more on that later).

Gouvea defines p-adic, and r-overconvergent, modular forms of level N and weight k via Katz' method: rules on (E, ω, P, Y) with E is an elliptic curve over a p-adically complete ring, P a level structure, ω a non-vanishing differential and Y such that $Y.E_{p-1}(E, \omega) = r$, such that $f(E, \lambda\omega, P, \lambda^{1-p}Y) = \lambda^{-k} f(E, \omega, P, Y)$ for all λ invertible on the base. Note that λ is an element of a rather general base, so you need k to be an integer; for κ a general weight one can't evaluate $\kappa(\lambda)$, so it's hard to make sense of this and we run into the age-old problem of how to give a conceptual definition of an overconvergent weight κ form.

Remark: Gouvea's notation is as follows. $F(B, k, N)$ is classical forms of level N weight k defined over the ring B, but allow poles at cusps. $M(B, k, N)$ is holomorphic even at cusps. Sans Serif F and M are p-adic or overconvergent versions of the same thing. Just as in Katz, the ON bases that Gouvea chooses for overconvergent forms of weight k are things of the form $f(r/E_{p-1})^a$, where f is classical of weight $k + a(p-1)$ with integral q-expansion.

2 I.3.

Something I know a bit less about is Katz' "generalised p-adic modular functions". Here is the idea: consider the scheme $Y_1(Np^m)$ over $\mathbf{Z}_p$ (note that this is a bit of a lousy name because it will be a rather non-standard model for the generic fibre) parametrising elliptic curves plus an embedding of μ_{Np^m}. This forces the point in characteristic p to be ordinary and furthermore one only gets one component in characteristic p as well, I guess. Let $X_1(Np^m)$ be its compactification. Let $W_{n,m}$ be the functions on $Y_1(Np^m)_{\mathbf{Z}/p^n\mathbf{Z}}$; now take the direct limit over the ms to get $W_{n,\infty}$ and then the projective limit over the ns to get W. Note that for $n = 1$ this is just going up the ordinary locus of the Igusa tower (the covering maps are etale, in fact). The "generic fibre of W" is morally the projective limit over m of the connected component of the ordinary locus of $Y_1(Np^m)$ that contains infinity. Note that we don't see overconvergent stuff here. Let V be the analogue of W but for X_1. A generalised p-adic modular function is an element of the ring W. Stuff in W still satisfies a q-expansion principle; the content of this is the irreducibility of the Igusa tower in characteristic p. The point is that W is representing the functor on p-adically complete rings sending B to elliptic curves over B equipped with (a point of order N and) a trivialisation of the formal group associated to E, that is, an isomorphism of this group with formal $\mathbf{G}_m$. Equivalently (formal groups are connected p-divisible groups) it's a compatible collection of embeddings of μ_{p^m} for all m. So this looks like a locally constant sheaf over the ordinary locus with fibre $\mathbf{Z}_p^\times$. Note that one can recover $V_{n,\infty}$ as V/p^n and then one can recover $V_{n,m}$ by taking invariants under $1+p^m\mathbf{Z}_p$.

Note that W and V have an action of $\mathbf{Z}_p^\times$, changing the trivialisation, so they are Λ-modules. More generally, one can work with schemes over B (some p-adically complete ring) instead of $\mathbf{Z}_p$ and get $W(B,N)$ and $V(B,N)$.

The funny thing is that a p-adic modular form of weight k can be thought of as an element of V; an element of V is just a rule for elliptic curves plus trivialisations, but to give a trivialisation is to give a differential (pull back the canonical one on formal $\mathbf{G}_m$) so weight k p-adic modular forms are just things in the $x \mapsto x^k$ eigenspace of V. More generally if κ is a character taking values in the p-adically complete ring R_0 and B is a p-adically complete R_0-algebra then one can think of weight κ p-adic modular forms over B as the κ-eigenspace of $V(B,N)$. Using this, Gouvea checks that for κ taking values in $\mathbf{Z}_p$, this definition coincides with Serre's.

3 I.3.6.

This section contains something I didn't know at all. If B is a p-adically complete DVR with field of fractions K then one defines the "divided congruences" $D_k(B,Np^m)$ as the subset of $\oplus_{j=0}^k M_k(\Gamma_1(Np^m);K)$ consisting of formal sums of forms which have q-expansions in $B[[q]]$. The point of course is that $(E_{p-1}-1)/p$ is in here if $k \geq p-1$. We let D be the limit of these divided congruence modules D_k; then D is a ring.

Lemma 3.1. *There's a natural injection from D into $V(B,N)$.*

Proof. Because D is a direct limit, an element of it is a finite sum. Multiply by it by a big power of p until it's the sum of modular forms with integral q-expansion. Now it's an element of V. But by the q-expansion principle we can divide by the big power of p. □

Lemma 3.2. *The image is dense.*

Proof. Wlog $m=0$. Check surjectivity mod p. But mod p we can "write down a basis" because Katz understands those etale covers $X_1(Np^r) \to X_1(N)$. □

Note that one can check easily that D is invariant under the Diamond ops. Note that all this works for cusp forms too; V_{par} is the cuspidal subspace of V, and the cuspidal subspace of D is dense in it.

4 I.3.7.

Back to stuff I knew already. Gouvea shows that Serre's p-adic modular forms of weight $\kappa : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ have a "modular" definition; they are rules on ordinary elliptic curves that transform via κ. Note that it *is* possible to compute $\kappa(\lambda)$ if λ is a unit in a complete p-adic ring; one defines $\kappa(\lambda)$ mod p^n for all n (by writing κ as a limit of integers k_n and computing λ^{k_N} instead for $N >> 0$) and then taking the limit. Gouvea even gives a definition of an r-overconvergent form of weight $\kappa : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ but can't prove anything about it. See p28 of his book.

5 Chapter II.

He does Hecke operators on V. Note that one can check that on the dense subspace D the Hecke operators have their usual definition! He does U and V but the results are weaker than Coleman's because he has the usual trouble with lifting the Hasse invariant and he also thinks a lot about his unit balls, which cause trouble too. He raises questions about norms of U on overconvergent forms, which Coleman theory doesn't really answer I guess. He shows that U has no spectral theory on p-adic forms (the neat construction of an eigenvector with eigenvalue λ for any λ non-unit) and proves U is compact on overconvergent forms, and does some spectral stuff a la Serre.

6 II.3.5.

He now says something about Hida theory. Note that we have got maps from p-adic, and hence overconvergent, forms of weight k, to $V(B, N)$. The amazing thing is that the ordinary projector can be defined on all of $V(B, N)$ at once, and one can prove that when restricted to overconvergent forms of weight k it gives the usual ordinary projector as defined via Fredholm theory or whatever. It's actually quite easy to see. We want to take the limit of $U^{n!}$ on $V(B, N)$. The reason this converges is that it converges on D_k which is a finite rank B-module, so there's $e : D_k \to D_k$ so there's $e : D \to D$, and D is dense in V, and now it extends to $V(B, N)$ by some kind of continuity that I don't quite follow. Hence e exists on p-adic modular forms of weight κ for all κ, in fact; well, that's using Gouvea's definintion. I guess one should check that E_κ is in $V(B, N)$ before one gets too carried away.

7 II.3.6 to II.3.7

He gets a lower bound for the NP under some assumptions. He conjectures that overconvergent forms of small slope are classical and also a weak form of Gouvea-Mazur (no precise bounds in either case).

8 Appendix II.4

He *states* that the Hecke algebra on $eV(B, N)$ is a finite flat Λ-module and that ordinary forms of weight $k \geq 3$ are classical.

9 Chapter III

He works out the duality: V_{par} is dual to a Hecke algebra T_0. So he can define a family of forms as a B-module homomorphism $T_0 \to B$. He writes down some big Galois representations. He notes that you can twist a p-adic modular form by a character κ (you add 2κ to the weight, as it were). I didn't know that! I guess it's kind of obvious when you think about it in terms of V.

10 Questions

Is the Eisenstein series E_κ in $V(B, 1)$?