Moduli spaces for abelian varieties over $\mathbf{C}$, II.

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Last modified May 2004. These are VERY SCRAPPY notes on the moduli problem in Carayol's paper, but over the complexes. I guess they were written to be read in conjunction with my publically available notes on abelian varieties, because, for example, I use notation from these other notes without any explanation.

1 Introduction

I vaguely knew the Hilbert and Siegel side of things, and even the indefinite quaternion algebra over $\mathbf{Q}$ story, when I started writing these notes on abelian varieties. But one thing I didn't know, and still don't really understand, is the case of a Shimura curve over a totally real field. Working out everything over the complexes will hopefully help my understanding of this. The plan: firstly I will write down some concrete cases of results of Shimura concerning other moduli problems—in particular I'll give far too many details about the story of endomorphisms by an imaginary quadratic field. This is a good example because it demonstrates the slightly unnerving phenomenon that if you force endomorphisms by one ring, you might implicitly be forcing endomorphisms by a bigger ring. I'll then talk about the degenerate case $F = \mathbf{Q}$ of Carayol's (really Deligne's, or Shimura's) moduli problem, and then perhaps move onto the general case of modèles étranges.[1] 1

2 Abelian varieties with an action of an imaginary quadratic field—new discrete invariants.

By "new" I mean "not discussed in previous sections", of course!

Let (V, L) be a g-dimensional piece of linear algebra data, and assume it's equipped with an action of an order in an imaginary quadratic field K. This induces an action of $K \otimes \mathbf{R}$ on V, and hence a second complex structure on V. One can think of the two complex structures as giving an action of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$ on V. V is now isomorphic to $\mathbf{C}^a \oplus \mathbf{C}^b$ as a module for $\mathbf{C} \oplus \mathbf{C}$. One might

[1] rewrite this as appropriate.

express an opinion as to whether to fix the values of a and b before we started doing any parameterising, as a and b are discrete invariants of this situation. This corresponds to what is probably our first real example of the observation that if we are parameterising polarizable (V, L) equipped with an action of some ring $\mathcal{O}$, then there is a chance that the induced analytic representation of $\mathcal{O}$ on V is not forced upon us. In all the cases we have considered so far, it has been the case that there was only one possibility for the analytic representation of $\mathcal{O}$ acting on V. But not any more, and this is surely the reason why in Deligne's "travaux de Shimura" he fixes the trace of the representation on the Lie algebra
of the abelian variety[2] Note that this phenomenon is also in Carayol: see the 2
formula $(*)$ on p166 of Carayol's "mauvaise reduction…" paper saying basically that we only parameterise abelian varieties with endomorphisms by some ring, such that the induced analytic representation has some fixed trace. I suspect that to give the representation is to give its trace in most or all of the situations that we consider.

3 Abelian surfaces with an action of an imaginary quadratic field.

We work out an explicit case of the above phenomenon. Choose an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-n})$. Let us consider an abelian surface up to isogeny, with an action of K, such that $a = b = 1$ in the notation above, and such that Rosati induces complex conjugation on K. Let's try and write down an explicit form for such an object. Shimura does this, and I was surprised by the answer. I was certainly surprised by the length of the argument, given that the real quadratic case "dropped out" in the Hilbert case in my earlier notes.

Let's fix the lattice L (or more precisely, the rational vector space $L_{\mathbf{Q}} := L \otimes \mathbf{Q}$) and vary the complex structure on V. We know that $L_{\mathbf{Q}}$ is a K-module and hence is free of rank 2. Choose a basis x_1, x_2. Let E denote the alternating form on $L \otimes \mathbf{Q}$ coming from the Riemann form. Then the map $K \to \mathbf{Q}$, $\lambda \mapsto E(\lambda x_i, x_j)$ is $\mathbf{Q}$-linear and hence is of the form $\lambda \mapsto \mathrm{tr}(t_{i,j}\lambda)$ for some $t_{i,j} \in K$. An easy calculation now shows that $T = (t_{i,j})$ satisfies $T = -\overline{T}^t$ and that E is the trace of the sesquilinear K-form on $K^2 = L \otimes \mathbf{Q}$ represented by T. Now $T\sqrt{-n}$ is Hermitian, K-valued, and sesquilinear, and hence can be diagonalised by a change of basis. So, changing our choice of x_1 and x_2 if necessary, we can assume $T = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)\sqrt{-n}$ with α and β rational. If T had a non-zero kernel then there would be non-zero w with $E(v, w) = 0$ for all v, contradiction, so $\alpha, \beta \neq 0$. We remark that we are still free to switch x_1 and x_2 around if necessary.

I find all this a bit confusing so let's imagine that we have chosen a lattice L in $L_{\mathbf{Q}}$, and $L \cong \mathcal{O}_K^2$, and x_1, x_2 are an $\mathcal{O}_K$-basis for L. Say $\{1, \sqrt{-n}\}$ is a $\mathbf{Z}$-basis for $\mathcal{O}_K$. Let's say that E is $\mathbf{Z}$-valued on L. This implies that all the $t_{i,j}$ are in the inverse different of $\mathcal{O}_K$. When one does the calculation, one finds that

[2] ref?

the statement that E is $\mathbf{Z}$-valued on L is just that $2\alpha n$ and $2\beta n$ are integers, and then the determinant of E on L is the square of $(2\alpha n)(2\beta n)$, up to sign. Hence one is forced to set $|\alpha| = |\beta| = 1/(2n)$ for this to be principal. But this is all assuming that L is free of rank 2 over $\mathcal{O}_K$, it's the usual confusion.

Let's go back to just thinking about things up to isogeny. It turns out that we can show that $\alpha\beta < 0$. Here's the proof, following Shimura.

We think of V as $L \otimes \mathbf{R}$. To give a complex structure on V is to give an action of i, that is, $G \in \mathrm{GL}_2(L \otimes \mathbf{R})$ with $G^2 = -1$ (Shimura writes H^t for G in his Annals paper "on analytic families of polarised abelian varieties and automorphic functions", and his definiteness is negative, not positive, by our conventions—but we'll stick with our conventions). To ensure that E is the imaginary part of a Hermitian form H on $L \otimes \mathbf{R}$ we must have $E(Gv, Gw) = E(v, w)$, that is, $G^t T \overline{G} = T$, and to ensure that H is positive definite we must have $G^t T \in \mathrm{GL}_2(L \otimes \mathbf{R})$ Hermitian and positive definite.

Note that x_1, x_2 is a K-basis for $L \otimes \mathbf{Q}$ and hence $x_1, \sqrt{-n}x_1, x_2, \sqrt{-n}x_2$ is a $\mathbf{Q}$-basis.

The one thing we haven't used yet is the fact that $a = b = 1$. So let's now assume we have a complex structure on V and let's diagonalise the K-action, that is, write $V = \mathbf{C}^2$ with $k \in K$ acting as k on the first factor and $\overline{k}$ on the second. With respect to this basis, let's say $x_i = (u_i, v_i)$. We deduce that a $\mathbf{Q}$-basis for $L \otimes \mathbf{Q}$ is (u_1, v_1), $\sqrt{-n}u_1, -\sqrt{-n}v_1)$, (u_2, v_2), $(\sqrt{-n}u_2, -\sqrt{-n}v_2)$. Hence an $\mathbf{R}$-basis for V is (u_1, v_1), $(iu_1, -iv_1)$, (u_2, v_2) and $(iu_2, -iv_2)$. The fact that these four vectors are linearly independent easily implies that there is no non-zero pair of complex numbers z_1, z_2 with $u_1 z_1 + u_2 z_2 = v_1 \overline{z_1} + v_2 \overline{z_2} = 0$, that is, that the matrix $X = \left(\begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix}\right)$ (those are complex conjugates, not fractions) is invertible. Hence if $u_1 = 0$ or $v_2 = 0$ then $u_2 v_1 \neq 0$ and by switching the order of the x_i we may assume WLOG that $u_1 v_2 \neq 0$.

We have now two descriptions of multiplication by $i \in \mathbf{C}$, one with respect to the K-basis and one with respect to the $\mathbf{C}$-basis. We unravel the underlying equation. Write $G = (g_{jk})$. We see $(iu_j, iv_j) = Gx_j = g_{1j}x_1 + g_{2j}x_2 = (g_{1j}u_1 + g_{2j}u_2, \overline{g_{1j}}v_1, \overline{g_{2j}}v_2)$ and hence $(iu_j, -i\overline{v_j}) = (g_{1j}u_1 + g_{2j}u_2, g_{1j}\overline{v_1}, g_{2j}\overline{v_2})$. That is, $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) X = XG$. Hence $G = X^{-1}\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) X$. The equations we are trying to solve are $G^t T \overline{G} = T$ and $G^t T$ Hermitian and positive definite. Substituting in for the first, we deduce that

$$X^t \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) X^{-t} T \overline{X}^{-1} \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right) \overline{X} = T$$

and hence that

$$\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) X^{-t} T \overline{X}^{-1} = X^{-t} T \overline{X}^{-1} \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right).$$

Hence $X^{-t} T \overline{X}^{-1}$ commutes with $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and this implies that it is a diagonal matrix! The second condition now becomes $X^t \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) X^{-t} T$ Hermitian, and hence $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) X^{-t} T \overline{X}^{-1}$ Hermitian and positive definite. We conclude that both conditions are satisfied iff $X^{-t} T \overline{X}^{-1} = \left(\begin{smallmatrix} -i\gamma & 0 \\ 0 & i\delta \end{smallmatrix}\right)$ with γ, δ positive reals. But we know $T = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)\sqrt{-n}$, and hence $\alpha\beta < 0$.

Hence the only possibilities for X are the elements of some twisted unitary group, essentially. Shimura now goes on to find a "normal form" for each such X (we can still change the elements of our $\mathbf{C}$-basis of $\mathbf{C}^2$ by non-zero complex factors, giving us some chance of normalising X somehow) and then works out which X give isomorphic abelian varieties—in other words he completely works out the nature of the moduli space (once one fixes T and so on; for the gory details see Shimura's paper). But I won't go into this because I don't need all the detail. Let's instead look more carefully at the statement

$$X^{-t}\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\sqrt{-n}\,\overline{X}^{-1} = \left(\begin{smallmatrix}-i\gamma & 0\\ 0 & i\delta\end{smallmatrix}\right).$$

This implies (taking the inverse of the LHS) that $\overline{X}\left(\begin{smallmatrix}\alpha' & 0\\ 0 & \beta'\end{smallmatrix}\right)\sqrt{-n}X^t$ is diagonal, for appropriate non-zero rationals α' and β'. Writing $X = \left(\begin{smallmatrix}u_1 & u_2\\ v_1 & v_2\end{smallmatrix}\right)$ and substituting in gives $\gamma' u_1 v_1 = u_2 v_2$ for some appropriate $\gamma' \in \mathbf{Q}^\times$.

The astonishing (to me) conclusion of this is that because $u_1, v_2 \neq 0$, we see that the matrix $M = \left(\begin{smallmatrix}0 & \gamma u_1/v_2\\ v_2/u_1 & 0\end{smallmatrix}\right) \in \mathrm{M}_2(\mathbf{C})$ is well-defined, not diagonal, and hence not in K, but sends x_1 to x_2, $\sqrt{-n}x_1$ to $-\sqrt{-n}x_2$, and so on—it preserves $L \otimes \mathbf{Q}$. Hence the endomorphism ring of any abelian variety with the properties we are considering is *necessarily* bigger than E. Note that the matrix above has square γ. One checks $\gamma = -\beta/\alpha$ is a positive rational, and of course conjugation by M induces complex conjugation on the K-action. Hence $\mathrm{End}^0(A)$ contains the quaternion algebra $\mathbf{Q} \oplus \mathbf{Q}i \oplus \mathbf{Q}j \oplus \mathbf{Q}k$ with $i^2 = \gamma > 0$, $j^2 = -n$ and $ij = -ji = k$ and so on and so on. Note that this quaternion algebra is indefinite as $\gamma > 0$.

Conclusion: Any polarized abelian surface with an action of K such that the polarization induces complex conjugation on K and such that the representation of K on the tangent space is $1 \oplus c$, is actually a false elliptic curve for some quaternion algebra split by K.

If I know there's a ppav in the isog class, can I say any more about the quat alg?

Shimura's article of course does much more: he says a lot about the situation for basically every possible endomorphism ring. for example if $a = 2$ and $b = 0$ then Shimura shows that the abelian surface is isogenous to the square of an elliptic curve with CM by K. So in fact there is no abelian surface A with $\mathrm{End}^0(A) = K$ (as Rosati is positive and hence would *have* to be complex conjugation).

Maybe I should also remark that the case we're really interested in is a generalisation of not quite the above calculations, but of the case of an abelian variety of dimension 4 equipped with an action of $\mathrm{M}_2(K)$, or a form of this ring. Shimura of course deals with this case too, and the calculations are essentially entirely the same. I kind of wish I'd typed them up too but the conclusions are the same: one first has to choose a T of exactly the same form as above—a two by two matrix blah blah, and one gets a moduli space parameterised by the open unit disc again.

4 Modéles étranges: the "ridicule" case.

We replace F by $\mathbf{Q}$ in the first few sections of Carayol's paper here, and try and see what's going on in this degenerate case (that Carayol in fact excludes—perhaps we will find out why.) Let's choose an indefinite quaternion algebra $B/\mathbf{Q}$, with discriminant d, and an imaginary quadratic field $E = \mathbf{Q}(\sqrt{-n})$. I had to make a hard choice here: I have consistently been using E to represent the alternating form—but Carayol uses ψ and I am going to be relying on his paper so much that I don't want to deviate at all from his notation.

Define $D = B \otimes E$. Then D is equipped with a natural involution, $h \mapsto \overline{h}$, the product of the canonical involution on B and complex conjugation on E. I had thought that if E splits B then after choosing an appropriate isomorphism $D \cong \mathrm{M}_2(E)$, this involution would be forced to be $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{smallmatrix}\right)$ but I realised later that this isn't the case: in fact, one can recover B knowing only D and the involution $h \mapsto \overline{h}$, because D is a quaternion algebra over E and hence has a canonical involution; B is thus the elements h of D whose canonical involution is $\overline{h}$.

Note that the involution $h \mapsto \overline{h}$ isn't positive: once one has tensored up to $\mathbf{R}$ then $h \mapsto \overline{h}$ really is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{smallmatrix}\right)$ if one chooses the isomorphism appropriately, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{smallmatrix}\right)$ might not have non-negative trace. We fix this thus: choose $\delta \in D^\times$ such that $\overline{\delta} = \delta$ and such that the involution $h \mapsto h^\iota := \delta^{-1}\overline{h}\delta$ is positive; such things exist in $D \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{C})$, for example $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, and hence also exist in D by a density argument.

Define t to be the map $t : D \to \mathbf{Q}$ which is simply the trace map $D \to E$ composed with the trace map $E \to \mathbf{Q}$. If E splits B then t is the trace of the following 4-dimensional representation of $B \otimes E$: let $B \otimes E \cong \mathrm{M}_2(E)$ act on E^2 considered as a 4-dimensional $\mathbf{Q}$-vector space. Then t is the trace of this representation. In general, choose a number field L disjoint from E that splits B; then $[LE : L] = 2$, $B \otimes LE \cong \mathrm{M}_2(LE)$ and t is the trace of the standard representation of $\mathrm{M}_2(LE)$, considered not as a 2-dimensional vector space over LE but as a 4-dimensional vector space over L.

What I tried to do first was the following: choose a maximal order $\mathcal{O}_D$ in D (note that D is a quaternion algebra over E so such things exist), fixed by ι. Now consider the problem of parameterising 4-dimensional abelian varieties $A/\mathbf{C}$ equipped with an action of $\mathcal{O}_D$, such that

(i) the trace of the induced analytic representation is t

(ii) the Rosati induces ι on D.

I had thought that this would give a decent moduli problem. But it doesn't. Somehow the problem is that we have no control whatsoever over the polarization: this control is somehow built into what looks like the level structure at the adelic level! On other cases there was a canonical polarization forced upon us, up to some reasonable equivalence, once we had fixed how Rosati acted on D. But not in this case. The adelic level structure that Carayol talks about, far from just being of the "point of order N" nature, is much more: the existence of a level structure of the kind Carayol asks for turns out to determine the "type" of the polarization to a large degree. The problem is that Shimura's classification of the objects above, even up to isogeny, involves an auxiliary choice of a matrix T, which represents the type of the polarization somehow (see the previous section for examples of the kind of thing that T is): one can think of $T\sqrt{-n}$ as an element of $\mathrm{M}_2(E)$ which is Hermitian and has negative determinant, and we care about the sesquilinear form that it represents. Unfortunately there are infinitely many classes of such forms giving us infinitely many components in the moduli space above.

So we really have to get our hands dirty now, and use adeles. Let's stick to the most degenerate case of all—$F = \mathbf{Q}$ and $B = \mathrm{M}_2(\mathbf{Q})$. Let's work out what Carayol's paper would say in this case—perhaps it will all still work. I remark again that Carayol explicitly excludes the case $F = \mathbf{Q}$.

Let B be $\mathrm{M}_2(\mathbf{Q})$, set $E = \mathbf{Q}(\sqrt{-n})$. Define $\alpha = \epsilon\sqrt{-n}$ with $\epsilon \in \mathbf{Q}^\times$. Note that D is canonically $\mathrm{M}_2(E)$ in this case. Define $\delta = \left(\begin{smallmatrix} 0 & \alpha \\ -\alpha & 0 \end{smallmatrix}\right)$. Then everything that Carayol demands on δ and α in his paper are true. Define a new involution on D: send h to $h^\iota := \delta^{-1}\overline{h}\delta$. This is the only time I'll deviate from Carayol's notation: I will use ι where he uses $*$. Working it all out explicitly, we see that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota = \left(\begin{smallmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{smallmatrix}\right)$—so ι is just conjugate-transpose. Now we define ψ on V, the $\mathbf{Q}$-vector space underlying D—well, let's differ from Carayol by a rational factor—let's let ψ be the map sending v, w to $\mathrm{tr}_{E/\mathbf{Q}}(vJw^\iota)$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It's now formal that $\psi(hv, w) = \psi(v, h^\iota w)$.

We have a $\mathbf{Q}$-vector space $V = D$, and a left action of D, and a $\mathbf{Q}$-linear alternating form. This gives us a reductive group G' over $\mathbf{Q}$ in the following manner—I will define the group as a functor—if R is a $\mathbf{Q}$-algebra, then $G'(R)$ is the R-linear maps $\beta : V \otimes R \to V \otimes R$ which commute with the D-action, and preserve the alternating R-form up to an element of $R^\times$, that is, such that $\psi(\beta v, \beta w) = \mu(\beta)\psi(v, w)$ for some $\mu(\beta) \in R^\times$. Note that μ is a morphism of functors $G' \to \mathbf{G}_m$. Now it's easy and formal that G' is representable—and it's also surely true that G' is reductive although I'm not sure why. So the Deligne machine applies if we have an h_0 satisfying all the axioms. We use the obvious one, in this case.

One can explicitly compute $G'(\mathbf{Q})$ because a $\mathbf{Q}$-linear map $V \to V$ that commutes with the left D-action sends 1 to some $h \in V$ and hence sends d to dh, and so is right multiplication by $h \in D^\times$. We'd like to think of this as a left action, so for $h \in D^\times$ let's consider the map $V \to V$ gotten by right multiplication by h^{-1}. With this convention, $G'(\mathbf{Q})$ naturally becomes a subgroup of $D^\times$, and, more generally, G' beomes a closed subgroup of the algebraic group represented by $D^\times$. An easy calculation shows that $G'(\mathbf{Q})$ is in fact strictly smaller than $D^\times$: it's the $h \in D^\times$ such that $h\overline{h}$ is rational. I guess that in general, $G'(R)$ is the h in $(D \otimes R)^\times$ with $h\overline{h}$ in $R^\times$.

Now let's work out more precisely what Carayol says is parameterised by the Shimura curve in this case, over $\mathbf{C}$. First a remark: if $A = V/L$ is an abelian variety, then its Tate module $T_l(A)$ is $L \otimes \mathbf{Z}_l$ and hence $\hat{V}(A) = L \otimes \mathbf{A}_f$ (of course this only depends on $L \otimes \mathbf{Q}$). Carayol wants to parameterise 4-dimensional abelian varieties A up to isogeny, equipped with an action of $\mathrm{M}_2(E)$, and such that

(i) the polarization induces ι on $\mathrm{M}_2(E)$,

(ii) the trace of the action on the tangent space is t,

and furthermore A is equipped with a class of D-linear symplectic similitudes $\hat{V}(A) \cong V \otimes \mathbf{A}_f$, defined up to an element of a compact open in $G'(\mathbf{A}_f)$.

So let's say $A = V/L$ is such an object. Shimura parameterises the A satisfying (i) and (ii), but his paramaterisation depends on the choice of some element $T \in \mathrm{M}_2(E)$ such that the polarization on A gives rise to the alternating form $v, w \mapsto \mathrm{tr}_{E/\mathbf{Q}}(\mathrm{tr}_{D/E}(vTw^\iota))$.

The key point is that for an arbitrary choice of T, there will be no D-linear symplectic similitudes $\hat{V}(A) \cong V \otimes \mathbf{A}_f$ at all! Let's explicitly see this. We have $\hat{V}(A) = L \otimes \mathbf{A}_f$ and so we are simply asking that, up to an element of $\mathbf{A}_f^\times$, the two alternating $\mathbf{A}_{f,E}$-valued forms on $V \otimes \mathbf{A}_f \cong \mathrm{M}_2(\mathbf{A}_{f,E})$ that we have, namely $v, w \mapsto \mathrm{tr}(vTw^\iota)$ and $v, w \mapsto \mathrm{tr}(vJw^\iota)$, are isomorphic. This means that T and $\alpha\delta$ give isomorphic sesquilinear forms over $\mathbf{A}_{f,E}$, up to an element of $\mathbf{A}_{f,E}^\times$. This is surely equivalent to the forms being isomorphic everywhere locally. So we are demanding that $T \in \mathrm{M}_2(E)$ satisfies $T^\iota = -T$ (this is a condition from Shimura's paper) and that, over $E \otimes \mathbf{Q}_p$ for any p, T is, up to a non-zero element of $E \otimes \mathbf{Q}_p$, of the form XJX^ι. This fact, and a matrix calculation, shows that if T is diagonalised (which is certainly possible) and $T = \left(\begin{smallmatrix} \gamma & 0 \\ 0 & \delta \end{smallmatrix}\right)$, with $\gamma, \delta \in \sqrt{-n}\mathbf{Q}$, then $-\gamma/\delta$ is locally a norm at all completions of E. But $E/\mathbf{Q}$ is cyclic and hence (see Tate's article in Cassels-Froehlich, section 9.6, for example) $-\gamma/\delta$ is a global norm. This means that T can be put into the form $\left(\begin{smallmatrix} \sqrt{-n} & 0 \\ 0 & -\sqrt{-n} \end{smallmatrix}\right)$. Now Shimura's machine completely characterises the 4-dimensional abelian varieties up to isogeny that can occur with this T, but I am getting weary, so perhaps I'll just say that we have a 4-d abelian variety with an action of $\mathrm{M}_2(E)$ so it's the square of a 2-d abelian variety with an action of E; we have just computed what T is, and so by the results in the previous section, the 2-d abelian variety also has another endomorphism i, whose square is, when you work it out, $+1$. Now $(1+i)/2)$ is an idempotent and hence even this surface is the square of an elliptic curve! So indeed we're parameterising elliptic curves.

Let's now do this more carefully. Let's choose a maximal order $\mathcal{O}_D$ in D—in fact, let's choose $\mathrm{M}_2(\mathcal{O}_E)$. Let's let α be $\sqrt{-n}$. Let $V_{\mathbf{Z}}$ denote the corresponding subspace of V. Then ψ is $\mathbf{Z}$-valued on $V_{\mathbf{Z}}$. Now $\mathcal{O}_{D_p}^\times$ will be compact, it being a closed subspace of a compact space. Hence there is a natural $G'(\mathbf{Z}_p)$ and the product of all these things could be our maximal compact K'. This is as "full" a level structure as we are ever going to get! If A is a 4-dimensional abelian variety with an action of $\mathcal{O}_D$, then to give a symplectic $\mathcal{O}_D$-linear isomorphism $\hat{T}(A) \cong V_{\hat{\mathbf{Z}}}$ modulo K' is just to state that such an isomorphism exists, which is just to state that if $A = V/L$ then L is everywhere locally isomorphic to $V_{\mathbf{Z}}$ as an $\mathcal{O}_D$-module plus pairing. I don't know whether this means that L is isomorphic to $\mathcal{O}_D$! But I would certainly believe that there would only be finitely many choices of isomorphism class for L.

Now let's consider the functor on p174 of Carayol's paper: we want to parameterise 4-dimensional abelian varieties equipped with an action of $\mathcal{O}_D$ such that the induced action on the tangent space has trace t, such that Rosati induces ι, and equipped with a symplectic $\mathcal{O}_D$-linear isomorphism $\hat{T}(A) \cong V_{\hat{\mathbf{Z}}}$ modulo K', i.e. such that L is everywhere locally isomorphic to $V_{\mathbf{Z}}$. Some components of this moduli space will comprise of the L which are actually isomorphic globally to $\mathcal{O}_D$. These objects are no doubt squares of abelian surfaces with an action of $\mathcal{O}_E$ but again we only get the element whose square is $+1$ in the endomorphism ring after a possible isogeny, so again we might have to take a component of the moduli space before we run across the surfaces which actually have endomorphisms by $\mathrm{M}_2(\mathbf{Z})$.

The conclusion: one component of Carayol's Shimura variety in this case is isomorphic to the j-line. The calculus of components is just a question of class groups that I'm not going to consider because at the end of the day Carayol only works with certain connected components.

The one thing I've not understood here is a formal construction: Shimura, if you follow his notes, constructs a moduli space for the above stuff as a finite union of quotients of the open unit disc by discrete groups coming from twisted unitary groups (well, I think that this might be what a twisted unitary group is). On the other hand, the standard models for modular curves all come from quotients of the upper half plane. As a concrete example, I think that the following must be true. Let Γ denote the subgroup of $\mathrm{GL}_2(\mathcal{O}_E)$ consisting of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with the property that $\gamma J \gamma^{\iota} = J$, where $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Define an action of Γ on the open unit disc: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = (\overline{a}z + \overline{b})/(\overline{c}z + \overline{d})$. Exercise: this is still in the unit disc. There is a bit of annoyance with units: if u is a unit then $\left(\begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix}\right) \in \Gamma$ and these things aren't acting trivially. Anyway, the quotient should be the j-line because it's parameterising a bunch of abelian surfaces with endomorphisms by $\mathrm{M}_2(\mathbf{Z})$.

Maybe I should work this out. The abelian surfaces in question are $\mathbf{C}^2/L$, with L the $\mathbf{Z}$-module spanned by $(1, z), (\sqrt{-n}, -\sqrt{-n}z), (z, 1), (\sqrt{-n}z, -\sqrt{-n})$. Then we note that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \sqrt{-n} & 0 \\ 0 & -\sqrt{-n} \end{smallmatrix}\right)$ both preserve L and indeed we can see that the automorphisms of L indeed include $\left(\begin{smallmatrix} a & b \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$ with $a\overline{a} - b\overline{b} = 1$. The question is: what's the corresponding elliptic curve? Well, looking at the image of $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ we realise that in fact we can see the elliptic curve $\mathbf{C}/L'$ where L' is the lattice spanned by $1 + z$ and $\sqrt{-n} - \sqrt{-n}z$. So how about sending the open unit disc to $\sqrt{-n}(1 - z)/(1 + z)$—this is in the upper half plane, assuming $\sqrt{-n}$ has positive imaginary part. Hey—an initial sketch seems to indicate that this works. I don't have the energy to run through the details but it does look fairly formal. Bingo.

5 Modèles étranges—comments on the general case.

This has nothing to do with the *arithmetic* in Carayol's thesis: I just want to understand the story over $\mathbf{C}$. I also confess that I am fast running out of time and energy for this project, so I'll be sketchy.

So let's choose a totally real field F distinct from $\mathbf{Q}$, and let's choose a quaternion algebra B/F, split at one infinite place τ_1 and ramified at all the others. Choose an imaginary quadratic field $K/\mathbf{Q}$, and set $E = FK$.

Following Carayol, define $D = B \otimes_F E$, a quaternion algebra centre E, possibly split—and in fact we could just choose K to ensure that it's split, as K and E only really play an auxiliary role in this whole thing. Now D comes equipped with the involution given by the canonical involution on B and complex conjugation on E. Let $h \mapsto \overline{h}$ denote this involution. Note that it's not positive because there are problems at τ_1. For any element $\delta \in D^\times$ with $\delta = \overline{\delta}$, we can define another involution by $h \mapsto \delta^{-1}\overline{h}\delta$.

Let V be D considered as a $\mathbf{Q}$-vector space. Here's an alternating form on V: firstly make another choice—choose $\alpha \in E^\times$ with $overline\alpha = -\alpha$ (for example, if $K = \mathbf{Q}(\sqrt{-n})$ just let $\alpha = \sqrt{-n}$) and define a pairing ψ on V by $\psi(v,w) = \mathrm{tr}_{E/\mathbf{Q}}(\alpha\,\mathrm{tr}_{D/E}(v\overline{w}\delta))$. It's elementary to check that ψ is alternating, and invertibility of α and δ implies that it's non-degenerate too. Note that $\psi(hv,w) = \psi(v,\delta^{-1}\overline{h}\delta w)$ too, but the involution $h \mapsto \delta^{-1}\overline{h}\delta$ won't be positive in general. The point is that we choose an appropriate δ so that it is.

If we consider K, α and δ as fixed, we now get a group G' of symplectic similitudes of (V,ψ), that is, elements of $D^\times$ such that $\psi(vl,wl) = \mu(l)\psi(v,w)$ where $\mu(l) \in \mathbf{Q}^\times$. One checks that $\mu(l) = l\overline{l}$ and hence $G'(\mathbf{Q})$ is in fact the elements $be \in B \otimes_F E$ with $\nu(b)e\overline{e} \in \mathbf{Q}^\times$, according to Carayol. Note that G' is in fact a reductive algebraic group, for general reasons. Note also that we are thinking of G' as acting on V acting on the left, as in the last section.

Carayol writes down a Hodge structure on $G'(\mathbf{R})$ and one can choose δ such that $h \mapsto \delta^{-1}\overline{h}\delta$ satisfies the usual positive-definiteness condition. Let's fix such a δ and let $h \mapsto h^\iota$ denote the corresponding involution.

For $h \in D$, define $t(h) \in \mathbf{Q}$ by firstly taking the trace down to EF, and then applying $\tau_1 + \overline{\tau_1} + 2\tau_2 + \ldots + 2\tau_g$, where the τ_i are the embeddings $F \to \mathbf{R}$, and we choose some fixed embedding $K \to \mathbf{C}$ to extend all the τ_i to E. Note that $t(1) = 4g$.

Now the big question is: what does Shimura say about parameterising abelian varieties of dimension $4g$ equipped with an action of D such that the induced analytic representation has trace t, and a polarization inducing ι on D? Shimura somehow gets a moduli space of dimension 1 again—the real places other than the first one contribute 0 to the dimension. Shimura's argument gives that if one chooses a T as above, then an explicit quotient of the open unit disc parameterises the abelian varieties of "type" T. There are a priori infinitely many choices for T, but in the moduli problem we demand conditions everywhere locally that will then cut the number of choices down to a finite number and the adelic machine will work.

I simply can't face typing Shimura's argument up, but I will end by giving the recipe. So in fact we have to choose T in D such that $T^\iota = -T$, and with some more conditions: if $1 \le \nu \le g$ then $\tau_\nu(T) \in \mathrm{M}_2(\mathbf{C})$ (with the isomorphism chosen so that ι is conjugate transpose) and we demand that $i\tau_\nu(T)^{-1}$ (which is now Hermitian symmetric) has signature $(1,1)$ if $\nu = 1$ and $(2,0)$ if $\nu > 1$. We choose witnesses to this fact, that is, matrices $W_\nu \in \mathrm{M}_2(\mathbf{C})$ with

$W_\nu(i\tau_\nu(T)^{-1})\overline{W}_\nu = \left(\begin{smallmatrix}1 & 0\\ 0 & \pm 1\end{smallmatrix}\right)$ where we choose the $-$ sign if $\nu = 1$ and the $+$ sign otherwise.

Now for any $z \in \mathbf{C}$ with $|z| < 1$ define $Y_1 = \left(\begin{smallmatrix}1 & z\\ \bar{z} & 1\end{smallmatrix}\right)$ and set $X_1 = Y_1\overline{W_1}$. Set $X_\nu = \overline{W_\nu}$ for $\nu > 1$. Now, unravelling Shimura, we write $X_1 = \left(\begin{smallmatrix}a_1 & b_1\\ c_1 & d_1\end{smallmatrix}\right)$ and $X_\nu = \left(\begin{smallmatrix}a_\nu & c_\nu\\ b_\nu & d_\nu\end{smallmatrix}\right)$ for $\nu > 1$, and write $x = (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, \ldots)$. Then $x \in \mathbf{C}^{4g}$ and, if we've normalised the action of D on $\mathbf{C}^{4g}$ explicitly the way Shimura does it, then Dx is the $\mathbf{Q}$-lattice we're after.

Ugh! See why Deligne just avoids these calculations at any cost!

I guess I have kind of run out of steam. On the other hand, I now feel prepared to read Carayol's thesis, in the sense that I am not so scared of the objects playing a starring role, so this write-up has done its job.