An example of a non-paritious Hilbert modular form.

Kevin Buzzard

April 26, 2012

1 Background and definitions.

[written Dec 2002]

A good source for notation and definitions is the paper [1]. We start by summarising some of this paper.

Let F be a totally real field of degree n over $\mathbf{Q}$, with integers $\mathcal{O}$. Note: Shimura uses some other funny letter, not $\mathcal{O}$.

Shimura firstly defines "classical" Hilbert modular forms of weight k, as functions on n copies of the upper half plane which transform well under elements of a congruence subgroup Γ of $\mathrm{GL}_2^+(F)$, the invertible matrices with totally positive determinant (see below for his definition of a congruence subgroup). His definition involves the $k/2$th power of a determinant; this is OK because his determinant is always totally positive and he takes the positive roots. In particular his centre always acts trivially. Let $\mathcal{M}_k(\Gamma)$ denote the space of such forms.

The definition of congruence subgroup that Shimura uses is: $\Gamma \subset \mathrm{GL}_2^+(F)$ is a congruence subgroup if it contains all the matrices in $\mathrm{SL}_2(\mathcal{O})$ congruent to the identity mod N for some ideal N of $\mathcal{O}$, and if furthermore the image of Γ in $\mathrm{PGL}_2(F)$ is commensurable with the image of $\mathrm{SL}_2(F)$. So there is a finite index subgroup of the totally positive units in $\mathcal{O}_F$ such that Γ contains $\mathrm{diag}(u, u^{-1})$ for all u in this subgroup. On the other hand we don't know much at all about the units u such that $\mathrm{diag}(u, u) \in \Gamma$; this could hold just for $u = 1$ or alternatively it could hold for any unit; both $\mathrm{SL}_2(\mathcal{O})$ and $\mathrm{GL}_2^+(\mathcal{O})$ are congruence subgroups. This is OK for him—the associated spaces of modular forms are the same—because his centre acts trivially.

Such Hilbert modular forms have a Fourier expansion $\sum_\xi c(\xi)e_F(\xi z)$ where ξ ranges over 0 and the totally positive elements of a lattice in F, $c(\xi)$ are complex numbers, and $e_F(z)$ is $e^{2\pi i \sum_\nu z_\nu}$. It is formal to check that $c(u^2\xi) = u^k c(\xi)$ if $\mathrm{diag}(u, u^{-1}) \in \Gamma$ and moreover that $c(u\xi) = u^{k/2}c(\xi)$ if $\mathrm{diag}(u, 1) \in \Gamma$ (note that this implies that u is totally positive).

Taking direct limits over all congruence subgroups Γ and get $\mathcal{M}_k$. Remark: this space is zero unless either all the k_i are non-negative and the same, or they are all positive. On the other hand there's no reason why they should have the same parity yet. Shimura firstly proves (Proposition 1.2) that if an element of $\mathrm{Aut}(\mathbf{C})$ acts on an element of $\mathcal{M}_k$ on coefficients, then the result is an element of $\mathcal{M}_{\sigma(k)}$. The proof is fairly short but does involve pulling back to Siegel modular forms. As a consequence of the proof one sees (Proposition 1.3) that for $f \in \mathcal{M}_k$, the field generated over $\mathbf{Q}$ by the $c(\xi)$, $\xi \neq 0$, must contain $c(0)$ and furthermore is finitely-generated over $\mathbf{Q}$. Now let N be the Galois closure of F and consider the weights as being indexed over maps $F \to N$. As a consequence of the above rationality result, Shimura shows in Proposition 1.7 that if Φ_k is the subfield of N corresponding to the subgroup of $\mathrm{Gal}(N/\mathbf{Q})$ consisting of elements which fix k, then $\mathcal{M}_k = \mathcal{M}_k(\Phi_k) \otimes_{\Phi_k} \mathbf{C}$ where $\mathcal{M}_k(R)$ denotes the subspace of $\mathcal{M}_k$ consisting of forms such that all $c(\xi)$ are in R (he only uses the cusp at infinity).

Shimura now takes the adelic point of view. He lets h denote the narrow class number of F and, for an integral ideal $c \subseteq \mathcal{O}$ and a character $\psi_0 : (\mathcal{O}/c)^\times \to \mathbf{C}^\times$, forms an adelic space of forms of level c, weight k and character ψ_0 explicitly as a direct sum of h of the above classical spaces. Note that he doesn't use Grössencharacters, for some reason. He calls this space $\mathcal{M}_k(c, \psi_0)$ and he notes that this space is identically zero unless $\psi_0(u) = \mathrm{sgn}(u)^k$ for every $u \in \mathcal{O}^\times$. He defines a Hecke action of $T(\mathbf{p})$ for $\mathbf{p}$ a maximal ideal of $\mathcal{O}$, and $S(\mathbf{p})$ for all $\mathbf{p}$ prime to c, on this space, and observes that if we have a non-zero eigenform for all $S(\mathbf{p})$ in this space, then there is a Grössencharacter explaining this, and extending ψ_0. He then splits things up into eigenspaces for each Grössencharacter ψ extending ψ_0, and calls the spaces $\mathcal{M}_k(c, \psi)$. The sum is over the characters ψ of finite order, conductor dividing c times the infinite places, that restrict to ψ_0 at the finite places, and which are sgn^k at infinity.

For $\mathbf{f} \in \mathcal{M}_k(c, \psi_0)$, by definition $\mathbf{f}$ has h q-expansions associated with it, and let the coefficients be called things like $a(\xi)$. For a non-zero integral ideal $n \subseteq \mathcal{O}$ he defines $c(n, \mathbf{f}) = a(\xi)\xi^{-k/2}$ for some appropriate ξ and he defines $C(n, \mathbf{f}) = c(n, \mathbf{f})N(n)^{k_{max}/2}$. If $c(\mathcal{O}, \mathbf{f}) = C(\mathcal{O}, \mathbf{f}) = 1$ then $\mathbf{f}$ is called normalised, and in this case it's $C(n, \mathbf{f})$ which is the eigenvalue of $T(n)$. Shimura defines

$$D(s, \mathbf{f}) = \sum_n C(n, \mathbf{f})N(n)^{-s},$$

the sum over all integral ideals of $\mathcal{O}$.

Note that we still haven't assumed any parity conditions on k. Shimura's Proposition 2.8 says that for a non-zero eigenform, **if** the parity condition is satisfied, then the field generated by the $C(n, \mathbf{f})$ has finite degree over $\mathbf{Q}$, and is totally real or CM. Shimura observes that this is not true in general without the parity condition. This is what I want to understand. Looking at (2.26) it seems to me that the Galois representation associated to f should have trace $C(\mathbf{p}, \mathbf{f})$ and determinant a finite order character multiplied by the $k_{max} - 1$th power of the cyclotomic character. Of course the point is that there shouldn't be a Galois representation when the parity condition isn't satisfied.

The first thing to do is to understand the functional equation for the L-

function of $\mathbf{f}$. If

$$R(s,\mathbf{f}) = N(c\delta^2)^{s/2}(2\pi)^{-ns}\prod_\nu \Gamma(s-(k_{max}-k_\nu)/2)D(s,\mathbf{f})$$

then the functional equation says that

$$R(s,\mathbf{f}) = i^{\sum_\nu k_\nu}R(k_{max}-s,\mathbf{f}|J_c)$$

where J_c is some funny thing, but $\mathbf{f}|J_c$ is some constant times the conjugate of $\mathbf{f}$, and in particular the constant, which will presumably be some kind of Gauss sum, will just show up as a constant in the functional equation independent of s.

We now compare this functional equation with the functional equation of a Hecke character, in the sense of Miyake's book on modular forms, theorem 3.3.1, and my notes on Grössencharacters, and ask ourselves when the L-function of $\mathbf{f}$ can possibly be the L-function of a Grössencharacter of a totally imaginary quadratic extension of F. The clue is the Γ factors. Note firstly that the factors of 2π seem to match up perfectly (note that the degree of the CM field is twice the degree of F) and the conductor parts look pretty close too. If ψ is the Grössencharacter such that $\Lambda(\psi,s) = R(s,\mathbf{f})$ then we must have $|\psi| = ||.||^\sigma$ with $2\sigma+1 = k_{max}$ (note that this looks right: if $k_{max}=1$ then we want the Grössencharacter to be unitary and algebraic). Now looking at the Γ factors gives us $s-\sigma+iv+\frac{1}{2}|u| = s-\frac{1}{2}(k_{max}-k)$ and hence $v=0$ and $(k_{max}-1-|u|)/2 = (k_{max}-k)/2$, so we see that we must have $k=1+|u|$. This looks good because all the ks should be ≥ 1.

Conclusion: if the L-function of a hilbert modular form equals the L-function of a Grössencharacter of a quadratic CM extension of F then we must, presumably, unless L-functions have more than one functional equation, which they don't, have, at each infinite place, $u=1-k$ or $k-1$, and $v=0$. Note that the sign of u shouldn't really be well-defined anyway, it depends on choices. An elementary calculation shows that the actual shape is as follows: at the complex infinite places of the CM field, the character should be of the form $x^{(k_{max}+k-2)/2}(cx)^{(k_{max}-k)/2}$. Note in particular that the Grossencharacter is algebraic iff the form is paritious.

2 An explicit example of a non-paritious form.

[written 2002, edited Nov 2009]

Let us then try and construct a Hilbert modular eigenform of weight (1,3), and also one of weight (1,2), for some real quadratic field. As ever, I'll take my favourite field $K := \mathbf{Q}(\zeta_8)$; magma tells me that K has trivial class group, its totally real subfield $F := \mathbf{Q}(\sqrt{2})$ has a non-totally-positive positive unit and trivial narrow class group; the integers of K are just $\mathbf{Z}[\zeta_8]$, and the unit group of K is $(\mathbf{Z}/8\mathbf{Z})\times\mathbf{Z}$, the group of order 8 being the roots of unity, and the $\mathbf{Z}$ being generated by $\sqrt{2}-1$. So the group $K^\times\backslash\mathbf{A}_K^\times$ is isomorphic to $U\backslash\left(\widehat{\mathcal{O}}_K^\times K_\infty^\times\right)$, the units embedded diagonally of course. Let the first complex place be the one sending ζ_8 to $e^{2\pi i/8}$, that is, "the identity", and let the other one be the one sending ζ_8 to ζ_8^3. We want a Grössencharacter which on the first complex place looks like $(x/|x|)^u$, with $|.|$ the usual norm on $\mathbf{C}$ (that is, the one involving square roots), and where $u = 1$ or 2, and which is trivial on the second complex place. Note that if $u = 2$ then the Grössencharacter will be algebraic, as at the first infinite place it's just $x/c(x)$, and the induction of the associated Galois representation will be a twist of the Galois representation associated to the weight $(1, 3)$ form; on the other hand if $u = 1$ then the Grössencharacter won't be algebraic.

We know what's going on at the infinite place now; so the game is to write down a finite order character on the finite places $\chi : \widehat{\mathcal{O}}_K^\times \to \mathbf{C}^\times$ such that $\chi(\zeta) = \zeta^{-u}$ for ζ one of the 8th roots of unity (embedded diagonally), and such that $\chi(\sqrt{2} - 1)$ is trivial (again embedded diagonally).

Maybe I should fix embeddings; let's imagine that $\zeta_8 = \exp(2\pi i/8)$ and $\sqrt{2} > 0$ in the complexes. Here's a nice prime of K to know about; it's $P = (41, \zeta_8 - 3)$; this lies above 41, which splits completely in K, being 1 mod 8. Note that the inverse of 3 mod 41 is 14 so (observing that $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$) we see that P lies above the prime $(41, \sqrt{2} - 17)$ of F. The terrific thing about P is that $\mathcal{O}_K/P$ is the field with 41 elements, and the image of u is 16, an element of order 5, and the eight roots of unity in K span a subgroup of order 8 (generated by 3), so they are completely independent in a sense. So define $\chi : (\mathcal{O}_K/P)^\times \to \mathbf{C}^\times$ sending 3 to ζ_8^{-1} and 16 to 1, and extend χ to a character of $\widehat{\mathcal{O}}_K^\times$ in the obvious way. This has the property we wanted above, for $u = 1$, and its square has the property for $u = 2$.

Now we're all set, as it were. Let ψ_0 be the character of $\widehat{\mathcal{O}} + K^\times K_\infty^\times$ which is χ at the finite places, $z/|z| = z/\sqrt{z.\overline{z}}$ at the first infinite place and trivial at the second. Then ψ_0 extends to a Grossencharacter, from the discussion above. Let's multiply this character by the square root of the norm character, to get a character ψ_2 which at the first infinite place sends z to z and at the second sends z to $|z| = \sqrt{z.\overline{z}}$.

Let ψ_3 be the square of ψ_2, so ψ_3 is z^2 at the first infinite place and $z.\overline{z}$ at the second.

There are forms of weight $(1, 2)$ and $(1, 3)$ associated with ψ_2 and ψ_3, at least conjecturally, because of Langlands-like conjectures, but in fact using converse theorems one can no doubt actually prove that such forms exist.

Let's work out what the L-function of ψ_2 looks like, by evaluating it at some prime ideals. Let's use geometric Frobenius because then things come out integrally. Let p be a prime congruent to 3 or 5 mod 8. Then p is inert in F and then splits in K. Say p is 3 mod 8. Then write $p = a^2 + 2b^2$ and the factorization of p in K is $p = (a + b\sqrt{-2})(a - b\sqrt{-2})$. Let's try and fathom out what ψ_2 does to something which is the reciprocal of a uniformiser at $(a+b\sqrt{-2})$ (and the identity everywhere else). Well, this isn't in $\widehat{\mathcal{O}}_K^\times.K_\infty^\times$, so we multiply by the global element $a + b\sqrt{-2}$ and now it is. So ψ_2 of this will be $\chi(a+b\sqrt{-2})$ multiplied by $(a+b\sqrt{-2})\sqrt{p}$. Similarly ψ_2 of the idele at $a-b\sqrt{-2}$

will be an 8th root of unity multiplied by $(a-b\sqrt{-2})\sqrt{p}$. So, for example, if $a \equiv 1 \bmod 41$ and $b \equiv 0 \bmod 41$, something which will happen infinitely often, then $C((p),\mathbf{f}) = 2a\sqrt{p}$, for $\mathbf{f}$ the Hilbert modular form associated to ψ_2, and $C((p),\mathbf{g}) = 2p(a^2-2b^2)$ if $\mathbf{g}$ is the HMF associated to ψ_3.

Something similar happens for $p \equiv 5 \bmod 8$; then $p = (a+bi)(a-bi)$ and again if $a-1$ and b both happen to be 0 mod 41 then $\psi_2(a+bi) = (a+bi)\sqrt{p}$ and so on, and $C((p),\mathbf{f}) = 2a\sqrt{p}$ and $C((p),\mathbf{g}) = 2(a^2-b^2)p$. Note that $\mathbf{f}$ now already definitely has the property that its C-values don't generate a finite extension of $\mathbf{Q}$.

On the other hand, what are the actual eigenvalues of the Hecke operators? Let's do $\mathbf{f}$ first. Again let's say $p \equiv 3 \bmod 8$ and $p = a^2+2b^2$ with $a-1, b \equiv 0 \bmod 41$. Then $C((p),\mathbf{f}) = 2a\sqrt{p}$ so $c((p),\mathbf{f}) = 2a\sqrt{p}/p^2$. I suspect that this value is related to $a(\xi)$ for $\xi = p(\sqrt{2}-1)/(2\sqrt{2})$; this is where the different comes in. I think $a(\xi) = 2a\sqrt{p}/p^2.\xi^{k/2} = 2a\sqrt{p}/p^2\sqrt{p}$ multiplied by some mess to do with $\sqrt{2}$s and it seems to me that the $\sqrt{p}$s cancel! So $a(\xi)$ does appear to be in some explicit finite extension of K.

3 An automorphic take on all of this.

Written Nov 2009. Let $D_{k,w}$ denote the discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ in Carayol's 1986 ENS paper. So here $k \geq 2$ and $w \in \mathbf{Z}$ are integers which are congruent mod 2, and the representation has central character t^{-w}. Hence the representation of the Weil group associated to $D_{k,w}$ by Langlands will be, when restricted to $\mathbf{C}^\times$, of the form $z \mapsto (z^{k-1}(z.\overline{z})^x, \overline{z}^{k-1}(z.overline z)^x$ with x chosen to make the determinant match up, so $k-1-2x = w$, so x is necessarily non-integral and $D_{k,w}$ is cohomological and C-algebraic but not L-algebraic.

Now let's try and figure out how things must be normalised for Shimura. He has a form of weight k (now a vector) but all our central characters must be the same on $\mathbf{R}_{>0}$ because a Grossencharacter on F will have the same derivative at both infinite places because of the unit. So this form of weight $(2,1)$ constructed previously will, up to twist, have one component at infinity C-algbraic and the other L-algebraic. At infinity it might for example look like $z \mapsto (z,\overline{z})$ at the "2" place and $z \mapsto ((z.\overline{z})^{1/2}, (z.\overline{z})^{1/2})$ at the other.

Note that this form is neither C-algebraic nor L-algebraic.

Now let's consider the restriction of this automorphic representation to the group G consisting of element of GL_2/F whose determinant actually lies in $\mathrm{GL}_1/\mathbf{Q}$. I believe that this group still admits Shimura varieties.

References

[1] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math Journal Vol 45 No 3 pp637–679 (1978).