The Robba Ring.

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Brief notes on the Robba Ring. Last modified Jan 2008.

1 Pre-Robba.

Let K be a finite extension of $\mathbf{Q}_p$. For r a positive rational number, consider the ring $B(r)$ of rigid functions on the rigid space over K defined by $p^{-r} \leq |z| < 1$. Note that this space is not an affioid because it's open at one end. Hence one might expect the ring $B(r)$ to have nasty properties. But in fact it's a "Bezout domain", or something. Here's a key construction, due to Lazard.

Theorem 1. *(Lazard) Let $r \geq m_1 > m_2 > m_3 > \ldots > m_n > \ldots$ be a decreasing sequence of positive rationals that tends to zero. For each n let $P_n(T)$ be a polynomial in $K[T]$ with the property that all its zeros have valuation m_n, and that $P_n(0) = 1$ (we normalise the valuation so that the valuation of a uniformiser is 1). Then there is a function in $B(r)$ with the property that the zeros of this function are precisely the union of the zeros of P_n.*

The theorem is proved in Lazard's 1962 IHES paper "Les zéros d'une fonction analytique d'une variable sur un corps valué complet". The result is false for $K = \mathbf{C}_p$! If the valuation is not discrete then the best that one can do is to find a function whose zero set contains the zeros of P_n for all n; Lazard also talks about this case. For non-discretely-valued fields K one basically needs K to be spherically complete. Lazard is a reference for all this.

Proof. It's not hard. Naively one might hope to simply write down $f = \prod_n P_n(T)$. The problem is that this might not converge for $|z| < 1$: for example if $P_n(T) = 1 + T + p^{-1}T^n$ then for any $|z| < 1$ we have $|P_n(z) - 1| = |z|$ for n sufficiently large, and hence $P_n(z) - 1$ is not tending to zero, so the product does not converge.

However it's possible to modify $P_n(T)$ so that it does converge. The theory of the newton polygon tells us that the coefficient of T^i in $P_n(T)$ has valuation at least $-i.m_n$. Write down the formal inverse of $P_n(T)$ in $K[[T]]$. This also has the same property: the coefficient of T^i has valuation at least $-i.m_n$ and (this is where we use discreteness of K) the first $N_n := [1/m_n]$ coefficients are in the integers of K. Truncate the reciprocal at that point (so it's in $\mathcal{O}[T]$); call the truncation T_n. Set $Q_n = P_n.T_n$. Now T_n has no zeros or poles in the open unit disc, so Q_n and P_n have the same divisor (in the open unit disc). However Q_n is of the form $1 + a_{N_n}T^{N_n} + \ldots$ and again newton polygon theory tells us that for $M \geq T_n$ the valuation of the coefficient of T^M is at least $-M.m_n$, so the valuation of $Q_n(z) - 1$ is at least $-N_n.m_n + N_n v(z) = N_n(v(z) - m_n)$ and this tends to infinity with n if $|z| < 1$, because $v(z) - m_n$ is eventually at least $v(z)/2 > 0$ and $N_n = [1/m_n]$ tends to infinity. We're done! $\square$

One can also "factorize" in the following way. Given a non-zero rigid function f on the open unit disc over a discretely-valued field, one could consider its divisor (f), check that it has the properties required of it in Theorem 1, and hence write $f = gu$ with g an infinite product of the form in the proof, and u a function which doesn't vanish anywhere. However consider the function $f(T) = \log(1 + T)$. Serre has observed that one can define $q = ((1 + T)^p - 1)/((1 + T) - 1)$ and

ϕ on $\mathbf{Z}_p[T]$ by $\phi(1+T) = (1+T)^p$, and if for $n \geq 0$ we set $P_n(T) = \phi^n(q)/p = \frac{(1+T)^{p^{n+1}}-1}{(1+T)^{p^n}-1}$ then an elementary exercise involving binomial coefficients shows that for $|z| < 1$ we have

$$\log(1+z) = z \prod_{n \geq 0} P_n(z).$$

On the other hand the algorithm above won't produce these P_ns; the P_n aren't in $\mathbf{Z}_p[T]$ so one has to take their reciprocal and modify by some fudge factor and so on, and one recovers an ugly expression for log instead of the beautiful one above. Note that the reason one doesn't need to modify anything is that for all z in the open unit disc the infinite product converges, because $P_n(T) = \frac{1}{p}(1 + (1+T)^{p^n} + (1+T)^{2p^n} + \ldots)$ (p terms) and each term tends to 1 as n gets big, so the average also tends to 1.

2 The Robba Ring.

The Robba ring over K (a finite extension of $\mathbf{Q}_p$) is the union for all $0 < r < 1$ (with $r \in p^{\mathbf{Q}}$) of the functions on the open annulus $r < |z| < 1$ (this annulus is defined over K). Explicitly, an element of R has a power series $\sum_{n \in \mathbf{Z}} a_n T^n$ with $a_n \in K$ and this power series must converge on some annulus $r < |z| < 1$.

It appears to be a theorem that R is a Bézout ring, which seems to mean that every finitely-generated ideal is principal. From the above arguments, it seems to me that a principal ideal might basically look like a sequence of zeros tending to the boundary, with the zeros on each circle $|z| = r$ all being defined over K (and hence there being at least the denominator of $\log_p(r)$ of them, as it were!). In particular we can write $\log(1+T)/T = ab$ with $a = P_0 P_2 P_4 \ldots$ and $b = P_1 P_3 P_5 \ldots$. Note that the function ϕ above still makes sense on the Robba ring, although it does not make sense on the functions on the annuli: if F converges on $r < |z| < 1$ then $\phi(F)$ might not converge on such a big annulus, it will only typically converge on a smaller one.

3 Kedlaya's theorem.

Kedlaya proved something about ϕ-modules over the Robba ring. A ϕ-module is M, a finite rank free R-module, plus $F : M \to M$ a semilinear map, that is, $F(rm) = \phi(r)F(m)$, such that the induced map $F : R \otimes_{R,\phi} M \to M$ is an isomorphism. Note that this is *not* an étale ϕ-module: étale means something else much more complicated.

In fact one of the problems is *defining* étale! The Robba ring has no natural integral structure. But it does have the following subring: Kedlaya calls it Γ_{con} and it's a discrete valuation ring: it's just the power series in the Robba ring whose coefficients are in $\mathcal{O}_K$! Explicitly, it's the things in the p-adic completion $\mathbf{A}_K$ of $\mathcal{O}_K[[T]][1/T]$ which converge on some annulus.

If K is a finite unramified extension of $\mathbf{Q}_p$ then Berger would refer to Γ_{con} as $\mathbf{A}_K^{\dagger}$.

If X is the localisation of $\mathcal{O}_K[[T]][1/T]$ at the maximal ideal (p) then $X \subset \Gamma_{con} \subset \mathbf{A}_K$, and all three of these things are DVRs with residue field $k((T))$, and X isn't complete, and neither is Γ_{con} but Γ_{con} is Henselian, and $\mathbf{A}_K$ is complete.

Kedlaya proves that any ϕ-module over the Robba ring has a filtration whose subquotients descend to $\Gamma_{con}[1/p]$, the field of fractions of Γ_{con}. Now over a DVR there's a Dieudonne-Manin theorem, so we get slopes. Kedlaya's theorem is that the subquotients of his filtration all descend, and each descended module has all slopes the same, and the slopes are strictly increasing [the smallest slope is the slope of the submodule].

The slopes arising in this way are called the slopes of the ϕ-module over the Robba ring.

Kedlaya defines étale to mean slope zero.

If K is a finite unramified extension of $\mathbf{Q}_p$ then it's trivial that étale ϕ-Γ-modules over the Robba ring are the same as Galois representations! Because that's how they're born.