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Introduction

John Gibbons just showed me how to solve the equation of a moving pendulum! I didn't know that one could solve this equation! Probably it has been known since Newton or something. John asked me how the analogous calculation went when gravity was imaginary :-/ and here's my attempt to work it out.

1 Revision: the real case.

Consider a swinging pendulum. Let θ be the angle from the vertical to the pendulum. Gravity exerts a force of mg downwards. The pendulum shaft swallows up some of this force ($mg\cos(\theta)$) and what is left contributes to acceleration, and we deduce

$$\ell\ddot{\theta} = -g\sin(\theta).$$

John apparently lives in a world where $g = \ell = 1$ (this is not actually what he said; he said that one could replace t by λt and then every dot introduces a new factor of λ so we can rescale) but let us not do this because it confuses me a bit (no intuition); let's set $K = -g/\ell$ and solve the equation

$$\ddot{\theta} = K\sin\theta.$$

We should remember that K is supposed to be a negative real number at this point. Multiply by $\dot{\theta}$ and integrate, getting

$$\frac{1}{2}\dot{\theta}^2 = -K\cos(\theta) + E$$

(which is just conservation of energy; E is something like energy divided by ml). In this real case we'll also let E be a real number. Note that the left hand side is always non-negative so we had better have $E \geq K$ (recall that $K < 0$) because otherwise the left hand side will always be negative whatever the value of θ. Note also that the degenerate case $E = K$ has only the solution $\theta = \dot{\theta} = 0$ so we may as well assume $E > K$.

The trick now is to set $z := e^{i\theta}$. Then $\dot{z} = i\dot{\theta}z$ so the conservation of energy equation becomes

$$\frac{1}{2}\left(\frac{\dot{z}}{iz}\right)^2 = -K((z + z^{-1})/2) + E$$

and clearing denominators gives

$$\dot{z}^2 = Kz^3 + Kz - 2Ez^2$$

which shows that the pair $(z, \dot{z})$ lives on an elliptic curve over the reals. This hasn't solved the equation yet though, because the statement "$(z, \dot{z})$ lives on an elliptic curve" is simply the statement that energy is conserved. We'll come back to this equation in a minute when I've worked out what the Weierstrass $\mathcal{P}$-function is.

2 Weierstrass $\mathcal{P}$.

Let Λ be a lattice in the complex numbers. Define

$$\mathcal{P}_\Lambda(z) = z^{-2} + \sum_{0\neq\lambda\in\Lambda}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

Recall that $\mathcal{P}$ is periodic wrt Λ, is even, has a double pole at a point in Λ, and its derivative $\mathcal{P}'_\Lambda(z) = -2\sum_{\lambda\in\Lambda}(z-\lambda)^{-3}$ vanishes at points of $\frac{1}{2}\Lambda$ that aren't in Λ. $\mathcal{P}$ scales like this: $\mathcal{P}_{\alpha\Lambda}(\alpha z) = \alpha^{-2}\mathcal{P}_\Lambda(z)$.

For $k \geq 4$ even set $G_k(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}$. The key equation is that if $Y = \mathcal{P}'_\Lambda(z)$ and $X = \mathcal{P}_\Lambda(z)$ then

$$Y^2 = 4X^3 - 60G_4(\Lambda)X - 140G_6(\Lambda).$$

3 Solving pendulums with Weierstrass $\mathcal{P}$.

We know

$$\dot{z}^2 = Kz^3 + Kz - 2Ez^2.$$

Let's first kill the coefficient of z^2 by replacing z by $Z := z - 2E/3K$ and we get

$$\dot{Z}^2 = KZ^3 + 0Z^2 + \cdots.$$

Next we need to scale. Set $X := (K/4)Z$, so $\dot{X} = (K/4)\dot{Z}$ and substitute in and multiply by a constant. We get

$$\dot{X}^2 = 4X^3 + (K^2/4 - E^2/3)X + (EK^2/24 - E^3/27)$$

and of course the point is that this now looks awfully like the equation involving the $\mathcal{P}$-function.

The discriminant of the cubic had better be non-zero, because otherwise it has repeated roots and isn't an elliptic curve at all. Bashing it out, it turns out that we want to avoid the case $E^2 = K^2$. Because $E > K$ is assumed, it turns out that for the cubic to have distinct roots we should hence assume $E \neq -K$. The physical reason this case is tricky is that if $E = -K$ then $\theta = \pi$ and $\dot{\theta} = 0$ is a solution to the equation, which corresponds to the pendulum sitting in the unstable equilibrium position, a case which we choose to avoid. Presumably if we start elsewhere then the pendulum slowly swings up to the unstable equilibrium point, corresponding to the solution to the equations going off to a cusp.

In fact one can now see the dichotomy. Recall $K < 0$ and $E > K$. If $E > -K$ then there is too much energy in the system and it goes around and around, and this corresponds to the cubic having three distinct roots. If however $K < E < -K$ then $E^2 < K^2$ and $K + E < 0$, we'll never make it to the top, and the cubic has only one real root. I don't know quite what the significance of the number of real roots of the cubic is, but it clearly has some kind of physical meaning.

In any case, if we choose a lattice Λ with $-60G_4(\Lambda)$ and $-140G_6(\Lambda)$ being those real numbers in the coefficients of the cubic then we have a solution $X = \mathcal{P}_\Lambda(t + z_0)$ and $\dot{X} = \mathcal{P}'_\Lambda(t + z_0)$ for any $z_0 \in \mathbf{C}$. For this to have real physical meaning we would like $|z| = 1$ and this forces $|X + E/6| = |K|/4$ which will be true for some $X = \mathcal{P}_\Lambda(z_0) \in \mathbf{C}$ and these will be the meaningful initial conditions.

Now finally we would like a mathematical explanation for the physically obvious fact that in every case apart from $E = -K$, the motion of the pendulum is periodic. This is because $G_4(\Lambda)$ and $G_6(\Lambda)$ are real, so the j-invariant of Λ is real, so Λ is homothetic to one of the form $\Lambda_\tau := \mathbf{Z} \oplus \mathbf{Z}\tau$ with τ in the upper half plane and with real part either 0 or 1/2 (I don't recall John mentioning the case of real part 1/2 but it can surely happen). In fact is it the case that the lattice is rectangular iff the real points of the curve are disconnected iff the cubic has three real roots iff there's enough energy in the system to get the pendulum to the top? This sounds like it might be right. No doubt John will know.

4 The complex case.

The reason I let K stay, and indeed be negative, was that John now wants me to set $K = i$ but I think that all the formal algebra I did remains valid, and it's only comments about $E > K$ and number of real roots that have to be changed/ignored. Note that if E is real then it seems to me that the curve is still defined over the reals and so again we'll get periodic motion.

If however E is complex then the j-invariant of the curve could well be complex. The question now is that we are given explicit values for $G_4(\Lambda) = 1/120 + E^2/180$ and $G_6(\Lambda) = E/3360 + E^3/3780$ and we are wondering whether Λ contains any non-zero real numbers—this would correspond to periodicity—or contains any numbers which are close to being real.