Representations of real reductive groups (informal course, May-July 2004.)

Kevin Buzzard

April 26, 2012

NOTE THAT THE STUFF I WRITE HERE ON LOCAL LANGLANDS FOR GL(2) OVER THE REALS HAS BEEN SUPERCEDED BY A NOTE IN ../jacquet_langlands.

These are notes from a course I once gave. I wrote the notes as I gave the course and I learnt the theory as I gave the course too, so there may be some confusion in some places as the theory sank in. Read at your own risk! Note also that I prove some lemmas but no theorems; I try to give references for theorems instead. My goal was more modest—I wanted to understand definitions rather than proofs!

Last modified Sept 2005.

Lecture 1.

Introduction

This course is supposed to be some kind of real analytic analogue to the course on representations of p-adic groups that I gave last year. One of the motivations of the p-adic course was so that I could try and understand the statement of the Local Langlands conjecture, which had been recently proved. The real case is much easier, so I believe, but I understand so much less about it that perhaps it will take me just as long to get my head around it.

Here is some background. If K is a finite extension of $\mathbf{Q}_p$, then there's something called an n-dimensional F-semi-simple Weil-Deligne representation of K, which is basically a representation of the Weil group of K, a big topological group closely related to the absolute Galois group of K. These objects are rather algebraic. There is also something called a smooth irreducible admissible representation of $\mathrm{GL}_n(K)$, which is a typically infinite-dimensional complex vector space equipped with a certain "continuous" action of $\mathrm{GL}_n(K)$. These objects are rather more analytic. The Local Langlands correspondence is some clever link between these two objects. For $\mathrm{GL}_n(K)$ it's a bijection, but it's a bit messier for a general reductive group because of a phenomenon called L-packets; given one representation of the Weil group I think that there might be in general more than one admissible representation corresponding to it. Try as I might, in fact, I don't think I ever found a precise statement of the Local Langlands conjecture for a general reductive group over a p-adic field. But there should be some kind of link there.

There is an analogous link on the real side too, which I believe can be proved by just writing down everything on both sides and matching them up. The phenomenon of L-indistinguishability also shows up in the real case, and we might see explicit examples of it later, if and when I find some.

The "algebraic" side of things is rather elementary and I'll get it out of the way in the first section. The analytic side of things is much more of a mystery to me and I'll say a few words about it because I don't know too much more about it (indeed, at the time of writing this introduction I don't know anything at all about it, to be honest). Let's just stick to $G := \mathrm{GL}_n(\mathbf{R})$ for the minute. One is interested in certain continuous actions of G on certain topological vector spaces, e.g. Banach spaces. Before I tell you what little I know about them, I'll mention the finite-dimensional case, which is much easier. The study of the finite-dimensional representations of G is well-known (not to me, but in general) and is strongly related to the study of the finite-dimensional representations of the Lie algebra $\mathfrak{g}$ of G. For infinite-dimensional representations of G there is also a link, but it's more complicated and I don't know anything much at all about it. The link is something like the following: if I have a unitary representation of G on a Hilbert space, I either look at the subspace of vectors that are infinitely differentiable for this representation, or the K-finite vectors, I'm not sure. The resulting space is much smaller than the Hilbert space, and the Lie algebra acts on this smaller space. I'm not really sure how this goes and I might fix this section up later when I am sure.[1] 1

Just as over a p-adic field there was the notion of an admissible representation, so there is such a notion here, but, perhaps surprisingly, at least one of the definitions of an admissible representation of G that I have read is not actually a representation of G at all, it's a representation of $\mathfrak{g}$, plus an action of $K := O(n)$, the maximal compact subgroup of G consisting of matrices X with $XX^t = 1$. I will give definitions later and hopefully show you some examples for $\mathrm{GL}_2(\mathbf{R})$. But let's do Weil groups first, because they are much easier.

1 Weil groups for the reals and the complexes.

The Weil group of a local field is supposed to relate to both the multiplicative group of the field, and also the absolute Galois group of the field. There is actually a purely axiomatic approach to the notion of the Weil group of a local (or indeed a global field). I won't follow it at all, I'll just give two concrete definitions:

(1) If $K \cong \mathbf{C}$ then define $W_K = K^\times$ (obvious topology).

[1] Perhaps rewrite this introduction when I understand what's happening?

(2) If $K \cong \mathbf{R}$ then define W_K to be the union of $\overline{K}^\times$ and $j\overline{K}^\times$, with the rules that $j^2 = -1$ and $jcj^{-1} = \overline{c}$ (one checks easily that these rules are enough to tell you how to multiply any two elements of W_K together). Note that in this case $\overline{K}^\times$ is a normal subgroup of W_K and the quotient group is $\mathrm{Gal}(\overline{K}/K)$, cyclic of order 2.

Note that in both cases there is an exact sequence

$$1 \to \overline{K}^\times \to W_K \to \mathrm{Gal}(\overline{K}/K) \to 1.$$

One key property that the Weil group of a local field K is supposed to have, at least if you know about the non-archimedean case, is that its abelianisation is supposed to come equipped with an isomorphism with $K^\times$. Let's just fix the isomorphism in case (2) above: one checks easily (several cases) that the commutator subgroup of W_K is things of the form $c/\overline{c}$ with $c \in \mathbf{C}^\times$, that is, the unit circle in $\mathbf{C}^\times$. The quotient is hence naturally isomorphic to the union of $\mathbf{R}_{>0}$ and $j\mathbf{R}_{>0}$, and the isomorphism from $\mathbf{R}^\times$ to this sends -1 to j and $x > 0$ to $\sqrt{x}$. The square root is for compatibility with the Weil group of $\mathbf{C}$. Read Tate's article in Corvalis for more details.

Let's define a norm $||w||$ on W_K. If $K = \mathbf{C}$ then $||w|| = w\overline{w}$. If $K = \mathbf{R}$ then $||w||$ is the same thing for $w \in \mathbf{C}^\times$, and $||j|| = 1$. Note that the norm is a continuous group homomorphism $W_K^{ab} \to \mathbf{R}_{>0}$ and hence a continuous group homomorphism $K^\times \to \mathbf{R}_{>0}$ which turns out to be the norm coming from measure theory (that is, the function telling you how much multiplication expands a Haar measure on the additive group K) in both cases. Note also that in both cases, the kernel of the norm map is compact. Note finally that $W_{\mathbf{C}} \subset W_{\mathbf{R}}$ and the norms agree.

By a *representation* of a Weil group we mean a continuous map into $\mathrm{GL}(V)$, with V a finite-dimensional complex vector space. We have lots of examples of 1-dimensional representations of Weil groups; if $s \in \mathbf{C}$ then consider $w \mapsto ||w||^s$ (note that if $r > 0$ is a positive real then $r^s := \exp(s\log(r))$ makes sense). In fact there aren't too many more.

Lemma 1. *(a) The only continuous group homomorphisms $\mathbf{R} \to \mathbf{C}^\times$ are those of the form $x \mapsto \exp(sx)$, with distinct s giving distinct homomorphisms.*

(a.5) The only continuous group homomorphisms from the unit circle $S := \{z \in \mathbf{C} : |z| = 1\}$ to $\mathbf{C}^\times$ are of the form $z \mapsto z^n$ for some $n \in \mathbf{Z}$, with distinct n giving distinct homomorphisms.

(b) The only continuous group homomorphisms $\mathbf{R}_{>0} \to \mathbf{C}^\times$ are those of the form $x \mapsto x^s := \exp(s\log(x))$ for $s \in \mathbf{C}^\times$, with distinct s giving distinct homomorpsisms.

(c) The only continuous group homomorphisms $\mathbf{R}^\times \to \mathbf{C}^\times$ are of the form $x \mapsto x^{-N}||x||^s$ for $s \in \mathbf{C}$ and $N \in \{0, 1\}$, and distinct pairs (N, s) give distinct homomorphisms.

(d) The only continuous group homomorphisms $\mathbf{C}^\times \to \mathbf{C}^\times$ are of the form $z \mapsto z^{-N}||z||^s$ with $N \in \mathbf{Z}$ and $s \in \mathbf{C}$, and distinct pairs (N, s) give distinct homomorphisms.

Proof. Distinct data giving distinct homomorphisms is easy: just divide one representation by the other and the result is supposed to be 1.

(a) It suffices to prove that every group homomorphism $\mathbf{R} \to \mathbf{C}^\times$ agrees with $x \mapsto \exp(sx)$ on a small disc in $\mathbf{R}$. Now choose an open disc centre 1 radius 1 say; the pre-image of this contains an open neighbourhood of zero, say $(-\delta, \delta)$, and we can cut along the non-positive real axis and define a log on $\mathbf{C}^\times$ now, which is injective. We deduce the existence of a continuous "additive" (wherever this makes sense) map $(-\delta, \delta) \to \mathbf{C}$. Now say $\delta/2$ is sent to z; by continuity we see that the map is just multiplication by $2z/\delta$ on $[-\delta/2, \delta/2]$ and this is enough.

(a.5) Precomposing with the map $\mathbf{R} \to S$ given by $r \mapsto \exp(ir)$ we see that we need to classify the continuous group homomorphisms $\mathbf{R} \to \mathbf{C}^\times$ with 2π in the kernel; by (a) we just need to find all s such that $\exp(2\pi s) = 1$, that is, such that $2\pi s = 2\pi i n$ for some $n \in \mathbf{Z}$. We deduce that $s = in$ and that the representation is $\exp(ir) \mapsto \exp(inr)$ so we are done.

(b) Take logs and it follows from (a).

(c) Follows from (b).

(d) As a topological group, $\mathbf{C}^\times$ is the unit circle times $\mathbf{R}_{>0}$. For $\mathbf{R}_{>0}$ use (b); for the unit circle use (a.5). $\square$

So we've now seen all the 1-dimensional representations of Weil groups. Tate slightly renormalises things—he says that if $K \cong \mathbf{C}$ then the 1-dimensional representations of W_K are all of the form $z \mapsto \sigma(z)^{-N}||z||^s$ with $\sigma : K \to \mathbf{C}$ an isomorphism, $N \geq 0$, and $s \in \mathbf{C}$, and the only times distinct data gives the same isomorphism is when $N = 0$ in which case we don't mind which σ we choose. Perhaps this normalisation is motivated by a study of L-functions and epsilon factors, but I don't think it will concern me for the moment.

Having seen all the 1-dimensional representations of Weil groups, we move on to the higher dimensional case. By standard arguments, any continuous irreducible finite-dimensional representation of $W_\mathbf{C}$ has an eigenvector and is hence 1-dimensional, so we deduce that we have now seen all the irreducible n-dimensional representations of $W_\mathbf{C}$ and hence all the semisimple n-dimensional representations of $W_\mathbf{C}$.

For $W_\mathbf{R}$ there are some irreducible 2-dimensional representations. The point is that if ρ is an irreducible representation of $W_\mathbf{R}$ of dimension greater than 1 then the restriction of ρ to $W_\mathbf{C}$ must have an eigenvector, and if it's v then v and jv span an invariant subspace, so the dimension of ρ is 2, and ρ is induced from a character of $W_\mathbf{C}$. The easiest way of seeing what's going on is to note that any character of $W_\mathbf{C}$ is of the form $z \mapsto \sigma(z)^N||z||^s$ with $N \in \mathbf{Z}_{\geq 0}$ and if you induce this 1-dimensional representation then you get a 2-dimensional representation which is irreducible if $N > 0$, and reducible if $N = 0$. Conclusion: the irreducible representations of $W_\mathbf{R}$ are 1-dimensional of the form $W_\mathbf{R}^{\mathrm{ab}} = \mathbf{R}^\times \to \mathbf{C}^\times$ via $z \mapsto z^{-N}||z||^s$ with $N \in \{0, 1\}$ and $s \in \mathbf{C}$, and 2-dimensional induced from a character $z \mapsto \sigma(z)^{-N}||z||^s$ on $W_\mathbf{C}$, with $N \in \mathbf{Z}_{>0}$ and $s \in \mathbf{C}$.

Say that a representation of a Weil group W_K is "of Galois type" if it has finite image. This is iff it factors through $\mathrm{Gal}(\overline{K}/K)$. I'm not entirely sure how important these things are but that's the definition.

The irreducible representations of Weil groups above have L-functions and ϵ constants. You can just define these things via a list: for example in the complex case define $L(\sigma(z)^{-N}||z||^s) = 2(2\pi)^{-s}\Gamma(s)$; in the real abelian case define $L(x^{-N}||x||^s) = \pi^{-s/2}\Gamma(s/2)$ and so on. I didn't get this far in the lecture. See Tate Corvalis for more information.

Lecture 2.

2 Lie algebras

A Lie algebra over a field k of characteristic zero is a vector space $\mathfrak{g}$ over k equipped with an alternating bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called "bracket" and satisfying the Jacobi identity

$$[v,[w,x]] + [w,[x,v]] + [x,[v,w]] = 0.$$

I'll give motivation for this funny axiom in a minute. Example: if R is an associative not-necessarily-commutative ring with a 1, equipped with a map $k \to R$ such that the image of k lands in the centre of R, then we call R an associative k-algebra. Then R is a k-vector space and one can define $[a,b] = ab - ba$; this works and gives us a functor from associative algebras to Lie algebras. Not every Lie algebra can be made in this way though e.g. $\mathbf{R}^3$ with cross product isn't of this form because it's centreless (there is no v such that $v \times w = 0$ for all w). Note that there was always this arbitrary choice of direction in the cross product and this is reflected in Lie algebras—given a Lie algebra, define its opposite by $[v,w]_{new} = -[v,w]$ and this is still a Lie algebra. It's isomorphic to the old one though, via the isomorphism $v \mapsto -v$.

Motivated by the associative algebra thing, we say that $v, w \in V$ commute if $[v,w] = 0$ and say that V is commutative if every two elements commute.

It's hard to write down a Lie algebra structure on a vector space by defining $[e_i, e_j]$ for e_i a basis, because you have to make sure that the Jacobi identity holds.

Important example of a Lie algebra is $\mathfrak{gl}_n(k)$, the n by n matrices over k, with the bracket coming from the associative algebra structure on these matrices, that is to say, define $[X,Y] = XY - YX$. More generally if V is a vector space then $\mathfrak{gl}(V)$ is the ring of endomorphisms of V equipped with its Lie algebra structure.

A morphism of Lie algebras is a bracket-preserving linear map. Example: if $\mathfrak{g}$ is a Lie algebra then the map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $g \mapsto (h \mapsto [g,h])$ is a morphism of Lie algebras precisely because of the Jacobi identity.

There are two kinds of subobjects, as in ring theory—a Lie subalgebra is $\mathfrak{h} \subseteq \mathfrak{g}$ (remark: this says $h \subseteq g$ in German) such that $[h_1, h_2] \in \mathfrak{h}$ for all $h_1, h_2 \in \mathfrak{h}$. An ideal is $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[h,g] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$. You can quotient out a Lie algebra by an ideal and get a quotient Lie algebra. The kernel of a morphism of Lie algebras is an ideal.

Example: $\mathfrak{sl}_n(k)$, the trace zero matrices, is a Lie subalgebra of $\mathfrak{gl}_n(k)$.

Recall the functor from associative algebras (with a 1) to Lie algebras. This functor has an adjoint—the construction is called the universal enveloping algebra, and the way it works is this: if $\mathfrak{g}$ is a Lie algebra then define $\mathfrak{g}^n := \mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}$ (n times) (so $\mathfrak{g}^0 = k$ for example) and define an associative algebra $T := k \oplus \mathfrak{g} \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3 \ldots$, the product being tensor multiplication (note that this is not commutative, this is isomorphic to the non-commutative polynomial ring over k generated by a basis of $\mathfrak{g}$); this is called the tensor algebra of $\mathfrak{g}$. Then quotient out by the bi-ideal J generated by $a \otimes b - b \otimes a - [a,b]$ for $a, b \in \mathfrak{g}$. Set $U(\mathfrak{g}) = T/J$. Then $U(\mathfrak{g})$ is an associative k-algebra so it has the structure of a Lie algebra, and the obvious map $\mathfrak{g} \to U(\mathfrak{g})$ is a Lie algebra homomorphism. Better—any map from $\mathfrak{g}$ to the Lie algebra underlying an associative algebra R extends uniquely to a ring homomorphism from $U(\mathfrak{g})$ to R, so in fact $U(\mathfrak{g})$ is an adjoint to the "forgetful" functor. Let's record this.

Proposition 2. *Let A be an associative algebra. let τ be a linear map $\mathfrak{g} \to A$ such that $\tau([v,w]) = \tau(v)\tau(w) - \tau(w)\tau(v)$ for $v, w \in \mathfrak{g}$. Then τ extends uniquely to a k-algebra homomorphism $U(\mathfrak{g}) \to A$ (sending 1 to 1).*

A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ and the proposition above gives a bijection between the representations of $\mathfrak{g}$ into V and the associative algebra homomorphisms $U(\mathfrak{g}) \to \mathfrak{gl}(V)$. Also note that given a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, compose with the canonical $\mathfrak{h} \to U(\mathfrak{h})$ and get $\mathfrak{g} \to U(\mathfrak{h})$ and hence $U(\mathfrak{g}) \to U(\mathfrak{h})$ and this map is called $U(\phi)$.

$U(\mathfrak{g})$ is called the universal enveloping algebra of $\mathfrak{g}$. Its centre is denoted $Z(\mathfrak{g})$.

We really need to understand these objects. I don't think we'll ever take the enveloping algebra of an infinite-dimensional Lie algebra. So let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Is $U(\mathfrak{g})$ finite-dimensional? No, not in general. So how do we understand $U(\mathfrak{g})$? Is the map $\mathfrak{g} \to U(\mathfrak{g})$ injective even?

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Fix a basis $x_1, x_2, \ldots, x_n$ of $\mathfrak{g}$. Let y_i denote the image of x_i in $U(\mathfrak{g})$.

Theorem 3 (Poincaré, Birkhoff, Witt)**.** *The set $y_1^{e_1} y_2^{e_2} \ldots y_n^{e_n}$ with $e_i \in \mathbf{Z}_{\geq 0}$, form a basis for $U(\mathfrak{g})$.*

Proof. A couple of pages of messy algebra, one key point being that if $r \geq 0$ and $U_r(\mathfrak{g})$ denotes the image in $U(\mathfrak{g})$ of $\oplus_{n=0}^{r} \mathfrak{g}^r$, and if $s \geq 1$ and $v_1, \ldots, v_s \in \mathfrak{g}$ and π is a permutation of $\{1, 2, \ldots, s\}$, and $\sigma : \mathfrak{g} \to U(\mathfrak{g})$ is the obvious map, then $\sigma(v_1)\sigma(v_2)\ldots\sigma(v_s) - \sigma(v_{\pi 1})\sigma(v_{\pi 2})\ldots\sigma(v_{\pi s}) \in U_{s-1}$, which gives us spanning. The messy part is injectivity and this is done by repeated applications of the universal property. See Theorem 2.1.11 of Dixmier for the details. □

Corollary 4. *The map $\mathfrak{g} \to U(\mathfrak{g})$ is injective.*

A natural example of Lie algebras, indeed basically the only examples we're interested in, comes from the tangent space at the identity of a Lie group. I'll say more about this when I need it. The Langlands programme is all about representations of reductive groups over global fields, but before I make an attempt to understand the global theory I want to understand the local theory. This course is about what's happening at infinity. The tangent space of a reductive Lie group is a reductive Lie algebra so I'll now tell you what a reductive Lie algebra is. Firstly some basic definitions.

Let $\mathfrak{g}$ be a Lie algebra. If V and W are subspaces of $\mathfrak{g}$ then define $[V, W]$ to be the subspace generated by $[v, w]$ for $v \in V$ and $w \in W$.

Define $C^1(\mathfrak{g}) = \mathfrak{g}$, $C^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, $C^{n+1}(\mathfrak{g}) = [\mathfrak{g}, C^n\mathfrak{g}]$. This is the descending central series of $\mathfrak{g}$ and it's a decreasing series of ideals. A Lie algebra is nilpotent if the descending central series reaches 0.

Define $D^0(\mathfrak{g}) = \mathfrak{g}$, $D^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and $D^{n+1}(\mathfrak{g}) = [D^n\mathfrak{g}, D^n\mathfrak{g}]$. This is the derived series of $\mathfrak{g}$ and it's a decreasing series of ideals. A Lie algebra is solvable if the derived series reaches 0. Nilpotent implies solvable. Solvable Lie algebras don't have a very interesting representation theory: if k is algebraically closed then the only irreducible finite-dimensional representations of a solvable Lie algebra are 1-dimensional (Cor 1.3.13 of Dixmier).

There is a maximal solvable ideal of a finite-dimensional Lie algebra; this is called the *radical* of the Lie algebra. A Lie algebra is called *semisimple* if its radical is zero. A Lie algebra is called *reductive* if it is the direct sum of a semisimple Lie algebra and a commutative Lie algebra. I don't want to develop the general theory of reductive and semisimple Lie algebras so I will just give references when necessary. I found Dixmier and Serre's book both very useful. As an exercise check that $\mathfrak{sl}_2(k)$ is semisimple and hence $\mathfrak{gl}_2(k)$ is reductive.

Lecture 3.

Today, let all Lie algebras be finite-dimensional. Recall: a semisimple Lie algebra is a finite-dimensional Lie algebra which has no non-zero solvable ideals. Given any Lie algebra $\mathfrak{g}$ there's a maximal solvable ideal, called the radical of $\mathfrak{g}$, and if you quotient out $\mathfrak{g}$ by its radical then you get a semisimple Lie algebra.

A *simple* Lie algebra is a finite-dimensional Lie algebra $\mathfrak{g}$ with $\dim(\mathfrak{g}) > 1$ and such that the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$. One checks easily that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ and hence simple Lie algebras are semi-simple. More generally, a finite direct sum of simple Lie algebra is semi-simple. A little deeper is the converse:

Proposition 5. *A semisimple Lie algebra is a direct sum of simple Lie algebras.*

Proof. Dixmier Theorem 1.5.12. The proof isn't very deep but it does use the Killing form which is something I haven't introduced. The point is that there's a canonical bilinear form on a Lie algebra called the Killing form, and the Killing form on a semi-simple Lie algebra is non-degenerate (this in fact classifies semisimple Lie algebras) and you can use this form to find complements to ideals. □

As a consequence of this result one can in fact see that if $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \ldots \oplus \mathfrak{a}_r$ is a direct sum of the simple Lie algebras $\mathfrak{a}_i$ then the ideals of $\mathfrak{g}$ are just direct sums of some of the $\mathfrak{a}_i$. As another—every reductive Lie algebra is a direct sum of finite-dimensional commutative Lie algebras and simple Lie algebras. And another—the centre of a reductive Lie algebra is exactly its radical.[2] 2

Given a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we say that $W \subseteq V$ is *stable* if $\rho(g)w \in W$ for all $g \in \mathfrak{g}$ and $w \in W$. We say that ρ is *irreducible* if $V \neq 0$ and the only stable subspaes of V are 0 and V. A representation is said to be *completely reducible* if it's a direct sum of irreducible representations. This is not always the case; for example if $\mathfrak{g}$ is the upper triangular 2 by 2 matrices in $M_2(k)$ then the natural 2-dimensional representation isn't completely reducible, it has precisely three stable submodules, one each of dimensions 0, 1, and 2, and in particular the 1-dimensional stable submodule has no complement. Note also that in this case $\mathfrak{g}$ is solvable and not semi-simple. But things are much better in the semi-simple case:

Proposition 6. *A finite-dimensional representation of a semi-simple Lie algebra is completely reducible.*

Proof: Dixmier Theorem 1.6.3.

I'll now justify a claim I made last time: $\mathfrak{gl}_2(k)$ is reductive. First note that it's the direct sum of the scalars and the algebra $\mathfrak{sl}_2(k)$ of traceless matrices, so it suffices to prove that $\mathfrak{sl}_2(k)$ is semisimple. It suffices to prove that it's simple, so it suffices to prove that $[\mathfrak{g}, v] = \mathfrak{g}$ for any $0 \neq v \in \mathfrak{g}$. Let's write down a basis first: set $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $[h, e] = 2e$ and $[h, f] = -2f$ and $[e, f] = h$ and playing around, it all comes out. I'll now tell you the finite-dimensional representation theory of $\mathfrak{sl}_2(k)$ explicitly: given any integer $r \geq 0$ define ρ_r to be the $r+1$-dimensional representation given by

$$\rho_r(h) = \begin{pmatrix} r & 0 & 0 & \dots & 0 \\ 0 & r-2 & 0 & \dots & 0 \\ 0 & 0 & r-4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -r \end{pmatrix}$$

and

$$\rho_r(f) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

and

$$\rho_r(e) = \begin{pmatrix} 0 & \mu_1 & 0 & \dots & 0 \\ 0 & 0 & \mu_2 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mu_r \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

[2]Presumably it's true that an irred repn of $\mathfrak{g} \oplus \mathfrak{h}$ is just a repn of $\mathfrak{g}$ and a repn of $\mathfrak{h}$? I should find a reference for this if it's true (or prove it myself)

with $\mu_i = i(r-i+1)$. Dull exercise: this is a representation of $\mathfrak{sl}_2(k)$. Easy: it's irreducible (use f to show that e_{r+1} is in any invariant subspace; then use e to show that all basis vectors are). Trickier is

Proposition 7. *These are all the irreducible representations of* $\mathfrak{sl}_2(k)$.

Proof. See Theorem 1.8.4 of Dixmier. I'll sketch the case where $k = \mathbf{C}$. Then if ρ is any r-dimensional representation of $\mathfrak{sl}_2(k)$, $\rho(h)$ has an eigenvector v with eigenvalue λ and now one checks easily that for any $i \geq 0$, $v_i := \rho(e)^i v$ satisfies $\rho(h)v_i = (\lambda + 2i)v_i$. Now $\rho(h)$ only has finitely many eigenvalues so $v_i = 0$ for i sufficiently large. WLOG $v \neq 0$ but $\rho(e)v = 0$. Now do the same with $\rho(f)$; set $w_i = \rho(f)^i v$ for $i \geq 0$ and note $\rho(h)w_i = (\lambda - 2i)w_i$ so the non-zero w_i are linearly independent and are easily checked to span an invariant subspace; so they're a basis. With respect to this basis one easily checks that $\rho(e)w_i = i(\lambda+1-i)w_{i-1}$ for $i \geq 1$ by induction on i. The final flourish: $w_{r+1} = 0$ so $\rho(e)w_{r+1} = 0$ so $(r+1)(\lambda - r)w_r = 0$ so $\lambda = r$.

The general case reduces to this case if you know enough about base extensions and so on.[3]

3

□

This example is a part of the general flavour of the finite-dimensional representation theory of simple Lie algebras; the irreducible finite-dimensional representations form a nice discrete set. The general story is that the finite-dimensional representations are parameterised by some subset of a lattice in a finite-dimensional vector space. I will explain the full story in the next lecture.

Classification of semisimple Lie Algebras.

Let V be a finite-dimensional real vector space. If $0 \neq \alpha \in V$ and we choose $\alpha^* \in V^*$ such that $\alpha^*(\alpha) = 2$ then the map $s_\alpha : V \to V$ defined by $s_\alpha(v) = v - \alpha^*(v)\alpha$ is a reflection; it fixes $\ker(\alpha^*)$ and sends α to $-\alpha$. Now let R be a finite subset of V such that R spans V. One checks easily that there is at most one α^* as above such that s_α fixes R (chap V lemma 1 of Serre Alg de Lie semisimple complexes, hitherto known as "Serre"). Hence the notation s_α isn't ambiguous.

A *root system* is a pair (V, R) with V a finite-dimensional real vector space, R a finite subset of V not containing zero and spanning V, and having the following property: for each $\alpha \in R$ there exists (a necessarily unique) α^* in V^* with $\alpha^*(\alpha) = 2$, $s_\alpha(R) = R$, and $\alpha^*(\beta) \in \mathbf{Z}$ for all $\alpha, \beta \in R$. You can replace this last condition with the condition that $s_\alpha(\beta) - \beta$ is an integer multiple of α; this is easier on the eye when we come to draw pictures later, because the s_α will be reflections.

If $\alpha \in R$ then $-\alpha \in R$ too. (V, R) is said to be *reduced* if $\pm\alpha = \mathbf{R}\alpha \cap R$ for all $\alpha \in R$.

[3]Should I say something about repns of reductive groups just being the repn of the semisimple bit and then a char? Was this in Fulton-Harris?

The rank of (V,R) is the dimension of V. Examples: in rank 1 the only examples are $---*-0-*-----$ ($A1$) and $---*-*-0-*-*---$. There are four reduced examples in rank 2, namely $A1 \times A1$, A_2, B_2 and G_2. [4] 4

The Weyl group of (V,R) is the subgroup W of $\mathrm{GL}(V)$ generated by the s_α; it's finite because it embeds in $\mathrm{Aut}(R)$. These groups tend not to be very interesting, he said guardedly. One useful fact is that there is a W-invariant inner product on V, because W is finite (take a random inner product and then average over W) and with respect to this inner product the s_α are orthogonal and hence reflections.

A *basis* for a root system is something slightly odd; it's $S \subseteq R$ such that S is a basis for V and that every $\beta \in R$ can be written as $\beta = \sum m_i \alpha_i$ with m_i integers, either all ≥ 0 or all ≤ 0. Example of G_2 (picture)[5]. Theorem: 5
a root system has a basis. Proof Serre Chap V Theorem 1. If S and S' are bases then there exists $w \in W$ such that $wS = S'$. Proof: Serre Theorem 2 Chap V. Once you've chosen a basis, you get a (multi) graph: there's a vertex for each edge, and you draw $4\cos^2(\phi)$ edges between a pair of vertices, where ϕ is the angle between the edges (this notion can be expressed properly! It's $\alpha^*(\beta)\beta^*(\alpha)$). This number is clearly in the set $\{0,1,2,3\}$. Different choices of bases give isomorphic multigraphs. One checks that the graph is connected iff the root system is irreducible.

If V can be written as $V_1 \oplus V_2$ with $R \subset V_1 \cup V_2$ then $R_1 := R \cap V_i$ is a root system in V_i and V is said to be the direct sum of the root systems (V_i, R_i). A root system is said to be irreducible if it's non-zero and can't be decomposed in this way. Every root system is a direct sum of irreducible ones (obvious) and this decomposition is unique (harder).

Theorem 8. *The only non-empty connected Coxeter graphs are A_n, B_n, D_n, G_2, F_4, E_6, E_7, E_8.*

A_n ($n \geq 1$) is $e_i - e_j$ ($i \neq j$) in the sum-zero hyperplane of $\mathbf{R}^{n+1}$. B_n in $\mathbf{R}^n$ is $\pm e_i$ and $\pm e_i \pm e_j$. Its dual is C_n [$2e_i$ not e_i]. If $n \geq 2$ then D_n is $\pm e_i \pm e_j$ with $i \neq j$ (this is $A_1 \times A_1$ and A_2 for $n = 2, 3$).

The union of B_n and C_n Unfortunately you can't reconstruct the root system from its Coxeter graph because there's a catch; you don't know the lengths of the roots. For example in $\mathbf{R}^3$ you have $B_3 = \pm e_i$ and $\pm e_i \pm e_j$ and you also have $\pm e_i \pm e_j$ and $\pm 2e_i$. But this is basically the only problem. This can be fixed by adding arrows to the Coxeter graphs which point towards the shorter roots when there are two or three lines between two vertices. The resulting diagrams, multigraphs with arrows on them, are called Dynkin diagrams. The Dynkin diagram determines the root system.

Given a semisimple Lie algebra $\mathfrak{g}$ over $\mathbf{C}$ there's a purely algebraic way of constructing a reduced root system $(V(\mathfrak{g}), R(\mathfrak{g}))$ from it. The Theorem is

[4] draw pictures of these using xfig and import them in postscript when you have an hour to waste; see pages V4 and V5 of Serr

[5] picture.

Theorem 9. *Two semi-simple Lie algebras with isomorphic root systems are isomorphic. Given a reduced root system there's a semisimple Lie algebra giving rise to it. The Lie algebra is simple iff the root system is irreducible.*

The construction: given a root system you build a semisimple Lie algebra.

Lecture 4

Summary of last lecture: every finite-dimensional Lie algebra has a solvable radical and the quotient is semi-simple. A semi-simple Lie algebra is the direct sum of simple ones. We can actually write down all the simple Lie algebras over $\mathbf{C}$; there are 4 families and 5 exceptional ones. Last time I gave them names: A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, and E_6, E_7, E_8, F_4, G_2, and this was proved by constructing a reduced root system from a semisimple Lie algebra and then classifying reduced root systems via some kind of combinatorics (Coxeter graphs). I didn't give the proof, which has, at its heart, the construction of a reduced root system from a semisimple complex Lie algebra, and the construction of a complex semisimple Lie algebra from a root system. The latter is done in the appendix to chapter VI of Serre's book on complex semisimple Lie algebras and isn't really anything that we shall use. But the construction of the root system from the Lie algebra is useful to us, because it will serve as an introduction to some other useful things. Before I do that I'll just tell you the dictionary for the families, for completeness.

Simple root systems: you can see section V.16 of Serre's book to see the explicit root systems A_n, B_n and so on, he does all of them, corresponding to the Coxeter graphs that I drew last time.

Now the list of the Lie algebras. Note first that if $\mathfrak{g}$ is a semisimple Lie algebra then the map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $g \mapsto (h \mapsto [g, h])$ is an injection because semisimple Lie algebras have no centre. In particular any semisimple Lie algebra is a subalgebra of a matrix algebra and hence you shouldn't be surprised to see the below constructions as subalgebras of matrix algebras.

If $m \geq 1$ then there's a Lie algebra called $\mathfrak{sl}_m$ and it's the subalgebra of $\mathfrak{gl}_m$ (the m by m matrices over $\mathbf{C}$) consisting of trace zero matrices; the bracket is given by $[X, Y] = XY - YX$. If $m \geq 2$ then it's simple.

If $m = 2n \geq 2$ is an even integer then there's a Lie algebra called $\mathfrak{sp}_m$ (warning: sometimes called $\mathfrak{sp}_n$) which is defined to be $\{X \in \mathfrak{gl}_n : X^t J + JX = 0\}$ where $J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$, and where again the bracket is given by $XY - YX$. Note that $\mathfrak{sp}_2 = \mathfrak{sl}_2$. If $m \geq 2$ is even then $\mathfrak{sp}_m$ is simple.

Finally if $n \geq 1$ then $\mathfrak{so}_n = \{X \in \mathfrak{gl}_n : X + X^t = 0\}$ again with the bracket given by $[X, Y] = XY - YX$. This takes a while to become simple. If $n = 1$ then it's zero. If $n = 2$ then it's abelian. Next $\mathfrak{so}_3 = \mathfrak{sl}_2$. Next $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ so it's semisimple but not simple. Now $\mathfrak{so}_n$ for $n \geq 5$ is simple but we've seen some before: in fact $\mathfrak{so}_5 = \mathfrak{sp}_4$ and $\mathfrak{so}_6 = \mathfrak{sl}_4$. But I've now written down all the isomorphisms between these gadgets. The dictionary is: A_n is $\mathfrak{sl}_{n+1}$ for $n \geq 1$, $B_n = \mathfrak{so}_{2n+1}$ (arrow left to right) for $n \geq 2$, $C_n = \mathfrak{sp}_{2n}$ for $n \geq 3$, $D_n = \mathfrak{so}_{2n}$ for $n \geq 4$. So there are all the families. And there are five more simple Lie algebras corresponding to E_6, E_7, E_8, F_4 and G_2. Their dimensions are 78, 133, 248, 52, 14 respectively.

Finally I want to explain the construction of a reduced root system from a semisimple Lie algebra. I need to tell you first about Cartan subalgebras. I'm not sure about the truth of things I am about to say over a general field so let the base be $\mathbf{C}$ for a while. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ then the normaliser of $\mathfrak{a}$ in $\mathfrak{g}$ is $\{g \in \mathfrak{g} : [g, a] \in \mathfrak{a} \text{ for all } a \in \mathfrak{a}\}$; it's the biggest subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$ and such that $\mathfrak{a}$ is an ideal in it. A *Cartan subalgebra* of $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ which equals its own normaliser.

Theorem 10. *Cartan subalgebras exist; and any two are isomorphic, in the sense that if* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *are Cartan subalgebras then there exists an automorphism of the Lie algebra* $\mathfrak{g}$ *sending* $\mathfrak{a}_1$ *isomorphically onto* $\mathfrak{a}_2$*. In particular all Cartan subalgebras have the same dimension, and this dimension is called the* rank *of* $\mathfrak{g}$.

Proof. Serre Chapter III Theorems 1 and 2. □

Theorem 11. *If* $\mathfrak{g}$ *is semisimple then its Cartan subalgebras are abelian.*

Proof. Serre Chapter III Theorem 3. □

Example: in $\mathfrak{sl}_n$ an example of a Cartan subalgebra is the diagonal matrices D, and if $\gamma \in \mathrm{GL}_n(\mathbf{C})$ then $\gamma D \gamma^{-1}$ are other examples.

Here's the construction. If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra then for $\alpha \in \mathfrak{h}^*$, the dual space to $\mathfrak{h}$, define $\mathfrak{g}^\alpha$ to be

$$\mathfrak{g}^\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

For example $\mathfrak{h} \subseteq \mathfrak{g}^0$ and conversely if $x \in \mathfrak{g}^0$ then $[x, h] = 0$ for all $h \in \mathfrak{h}$ so x is in the normaliser of $\mathfrak{h}$ so $x \in \mathfrak{h}$ by definition of Cartan subalgebra. So $\mathfrak{g}^0 = \mathfrak{h}$. Note in particular that this shows that $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$. A *root* of $\mathfrak{g}$ is $\alpha \in \mathfrak{h}^*$ with $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$. Let R denote the set of roots in $\mathfrak{h}^*$. The big theorems are:

Theorem 12. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha$ *and the* $\mathfrak{g}^\alpha$ *are 1-dimensional.*

Proof. Serre chapter VI Theorems 1 and 2. □

It's not quite true that $(\mathfrak{h}^*, R)$ is a reduced root system because $\mathfrak{h}^*$ is a complex vector space. There's a definition of a complex root system and it's the same as the definition of a root system but with complex vector spaces. One can prove that every complex root system is the base extension of a unique real root system. The dictionary the other way is to consider the real vector subspace spanned by the roots in a complex root system. Hence let V denote the real vector subspace of $\mathfrak{h}^*$ spanned by R.

Theorem 13. (V, R) *is a reduced root system. It's called the root system attached to the pair* $(\mathfrak{g}, \mathfrak{h})$.

Proof. Serre Chapter VI Theorem 2. □

So now you can see why the Lie algebra G_2 is 14-dimensional; the Cartan subalgebra is 2-dimensional and there are 12 roots. Note that the construction of the root system from the Lie algebra involved choosing a Cartan subalgebra; but two distinct choices give isomorphic root systems. Finally I remark that if we choose $\alpha \in R$ then $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] =: \mathfrak{h}_\alpha$ is a 1-dimensional subspace of $\mathfrak{h}$; there exists a unique $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$; now choose $0 \neq e_\alpha \in \mathfrak{g}^\alpha$. There exists a unique $f_\alpha \in \mathfrak{g}^{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ and then $e_\alpha, f_\alpha, h_\alpha$ span a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$.

See chapter 22 of Fulton and Harris for an amazing example of reverse engineering where they construct the Lie algebra G_2 knowing only its root system and a few more standard facts about the $\mathfrak{g}_\alpha$

While I'm here, let me tell you about Borel subalgebras. The reason they're called Borel subalgebras is that Borel proved the following. Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan. Let (V, R) be the associated reduced root system. Choose a basis for this root system. Then every root is either positive or negative. Let $\mathfrak{n}$ be the direct sum of the $\mathfrak{g}^\alpha$ for α running through the positive roots and set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Then $\mathfrak{b}$ is called the *Borel subgroup* of $\mathfrak{g}$ relative to $\mathfrak{h}$ and the basis S of R. Note that $\mathfrak{n}$ is nilpotent, $\mathfrak{b}$ is solvable (Theorem 4 of Chapter VI of Serre) and Borel's theorem about these things is that for any solvable subgroup of $\mathfrak{g}$, there's an automorphism of $\mathfrak{g}$ mapping it into $\mathfrak{b}$. In particular, $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$. A Borel subalgebra can just be defined then as a maximal solvable subalgebra of $\mathfrak{g}$ in fact: given such a thing there's a Cartan and a basis of the corresponding root system that gives this Borel.

Example of all this: $\mathfrak{sl}_{n+1}$. Then $\mathfrak{h}$ can be taken to be the diagonal matrices with trace zero, $\mathfrak{h}^*$ is naturally the quotient of $\mathbf{C}^{n+1}$ by $\mathbf{C}$ embedded diagonally, but we may as well think of it as the matrices in $\mathbf{C}^{n+1}$ with sum of the entries equal to zero. Now if $i \neq j$ then the matrix $E_{i,j}$ with a 1 in the (i, j)th entry and everywhere else zeroes is a basis for $\mathfrak{g}^\alpha$ with α defined as $\alpha(d_i) = d_i - d_j$. Explicit example: $\mathfrak{sl}_2$; the associated root system is just A_1, the line with two points on it, α and $-\alpha$, and there are two choices for bases, and the associated Borels are the upper triangular and lower triangular matrices. Slightly more general example is $\mathfrak{sl}_3$; then (picture of a hexagon with vertices labelled things
like $e_1 - e_2$ and so on[6]) there are 6 bases and you explicitly check that they 6
correspond to the 6 orderings of the basis vectors so the Borels are conjugates of the upper triangular matrices by permutations.

Finally I want to tell you all about finite-dimensional representations of complex semisimple Lie algebras. Let V be any complex vector space, not necessarily finite-dimensional, and say $\mathfrak{g}$ (finite-dimensional, semi-simple) acts on V, i.e., we have a map $\mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebras. Choose a Cartan $\mathfrak{h}$. If $\omega \in \mathfrak{h}^*$ then let $V^\omega = \{v \in V : \rho(h)v = \omega(h)v \text{ for all } h \in \mathfrak{h}\}$. It's a subspace of V; an element in V^ω has *weight* ω. The dimension of V^ω is the *multiplicity* of ω in V. If $V^\omega \neq 0$ then we say ω is a *weight* of V. One checks easily that if α is a root and $g \in \mathfrak{g}^\alpha$ then $\rho(g)V^\omega \subseteq V^{\omega+\alpha}$, so the direct sum $\bigoplus_{\omega \in h^*} V^\omega$ is a submodule of V preserved by $\mathfrak{g}$.

[6] picture

To get any further we have to choose a base for our root system. So choose one, call it S. Choose $0 \neq e_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in S$. If V is a representation of $\mathfrak{g}$ and $\omega \in \mathfrak{h}^*$ and $v \in V$ then we say that v is a *primitive element of weight* ω if $0 \neq v \in V^\omega$ and if $e_\alpha v = 0$ for all $\alpha \in S$. Example: in our representation of $\mathfrak{sl}_2$, if we choose $\mathfrak{h}$ to be the diagonal matrices and S to be the base corresponding to the upper triangular matrices then e is the e_α we're looking at and the kernel of the action of e is spanned by the basis matrix e_1 which has weight r. So in this case there's a unique weight for which there are primitive elements. Note also that this construction is a way of recovering the "parameter" r from the representation constructed last time. The theorem is

Theorem 14. *If V is a (possibly infinite-dimensional) irreducible $\mathfrak{g}$-module ($\mathfrak{g}$ finite-dimensional and semi-simple) containing a primitive element v of weight ω then ω is the only element of $\mathfrak{h}^*$ for which there are primitive elements and all the primitive elements are contained in $\mathbf{C}v$. We say that ω is* the dominant weight *of V. We have $V = \bigoplus_\pi V^\pi$ as π runs through the weights of V; the weights of V are all of the form $\omega - \sum_i m_i\alpha_i$ as α_i run through the basis and $m_i \in \mathbf{Z}_{\geq 0}$; all V^π are finite-dimensional (and V^ω is 1-dimensional). Furthermore for any $\omega \in \mathfrak{h}^*$ there exists a unique (up to isomorphism) irreducible $\mathfrak{g}$-module with dominant weight ω.*

Proof. Theorems 1 and 2 of Chapter VII of Serre. □

One can explicitly construct the irreducible $\mathfrak{sl}_2$-module with dominant weight ω. Firstly let's construct an infinite-dimensional representation of $\mathfrak{sl}_2$: we have a basis $v_n : n \geq 0$ and let's choose $r \in \mathbf{C}$; define $hv_n = (r - 2n)v_n$ and $ev_n = n(r + 1 - n)v_{n-1}$ and $fv_n = v_{n+1}$. This is the same formula as before. If $r \in \mathbf{Z}_{\geq 0}$ then this isn't irreducible because $ev_{r+1} = 0$ so $\oplus_{n \geq r+1}\mathbf{C}v_n$ is an invariant subspace. But if r isn't a non-negative integer then I claim this is irreducible: the point is that if W is an invariant subspace then W is in particular stable under h and hence if $W \neq V$ then W is contained in $\oplus_{n\geq 1}\mathbf{C}v_n$ but no non-zero subspace of this is going to be e-stable because ev_n is a non-zero multiple of v_{n-1} if $n \geq 1$. Note that the same argument shows that if $r \in \mathbf{Z}_{\geq 0}$ then the unique invariant subspace is $\oplus_{n\geq r+1}\mathbf{C}v_n$ and the quotient is irreducible and indeed the finite-dimensional representation we constructed before.

Back to the general case. We have now see that after one has chosen a Cartan and a basis, one gets a canonical bijection between $\mathfrak{h}^*$ and the set of irreducible representations of $\mathfrak{g}$ that have a dominant weight. The next important result is

Theorem 15. *If V is an irreducible finite-dimensional representation of complex semisimple $\mathfrak{g}$ then V has a primitive element and hence a (unique) dominant weight.*

Proof. Corollary to Proposition 3 of Serre chapter VII. □

So now we only need to work out which elements of $\mathfrak{h}^*$ correspond to the finite-dimensional representations, and we've worked out all the representation theory of semisimple Lie algebras. In the case of $\mathfrak{sl}_2$ we can read off the answer:

if α is our basis for A_1 and h_α is the corresponding element of $\mathfrak{h}$ then $\omega(h_\alpha) = r$ is a non-negative integer for the finite-dimensional case. The general case is the obvious generalisation:

Theorem 16. *If $\omega \in \mathfrak{h}^*$ then the irreducible module with dominant weight ω is finite-dimensional iff $\omega(h_\alpha) \in \mathbf{Z}_{\geq 0}$ for all $\alpha \in S$ (here S is the basis).*

Proof. Theorem 3 of Serre VII. $\square$

Lecture 5

3 Affine algebraic groups

2 weeks ago we saw that we could classify semi-simple Lie algebras, isomorphism classes of such being in one to one correspondence with isomorphism classes of reduced root systems. Last week we gave some kind of classification of representations of Lie algebras. This week I want to generalise these two things to, well, I could say Lie Groups, but there's no real need to go any differential geometry, so I will stick to algebraic geometry.

Let K be a field. An *affine algebraic group* over K is an affine variety G over K (that is, a reduced affine scheme of finite type over K, and note, some people want varieties to be connected but we don't), equipped with a K-point e, a morphism $\mu : G \times G \to G$ and $i : G \to G$ satisfying the group laws. Standard fact: G has a smooth point, so G is smooth. Morphisms of such are morphisms of algebraic varieties which are also group homomorphisms. Example: $G = \mathbf{A}^n$, $G = \mathrm{GL}(n)$ (note that this is an affine variety), $G = \mathbf{G}_m$, $G = \mathrm{SL}(n)$, $G = \mathrm{O}(n) = \{g \in \mathrm{GL}_n : gg^t = 1\}$, $G = \mathrm{SO}(n) = \mathrm{SL}(n) \cap \mathrm{O}(n)$, $G = \mathrm{Sp}(n) = \{g \in \mathrm{SL}(2n) : g^t J g = J$ $(J = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right))$. $\mathbf{H}^\times$ is an algebraic group over $\mathbf{R}$; it becomes isomorphic to $\mathrm{GL}(2)$ over $\mathbf{C}$ but isn't isomorphic to $\mathrm{GL}(2)$ over $\mathbf{R}$. It's the subset of $\mathbf{A}^4$ defined by $a^2 + b^2 + c^2 + d^2 \neq 0$. A torus is a group of the form $\mathbf{G}_m^n$ for some n. More subtle (but currently my favourite kind of algebraic group): if L/K is a degree 2 field extension and $w \in \mathrm{GL}_n(L)$ satisfies $w^c = w^t$ then there's a group $U(n)$ over K defined by $U(n) = \{g \in \mathrm{GL}_n(L) : gwg^{ct} = w\}$ (note that this is not a group over L because c isn't an L-morphism).

Note the usual story: I've written down lots of subgroups of $\mathrm{GL}(n)$ and the reason is (Borel Lin Alg Gps Proposition 1.10): if G is an affine algebraic group over K then there is a morphism of groups $G \to \mathrm{GL}(N)$ for some big N which induces an isomorphism between G and a closed subgroup of $\mathrm{GL}(N)$. (note this means closed in the Zariski topology, so in particular it means that the image is the set of zeroes of a bunch of polynomial equations).

If G is an affine algebraic group then its identity component G^0 is a subgroup, and the quotient is finite. We will frequently assume that G is connected (else we'd get all of finite group theory in our theory!).

Now let K be algebraically closed of characteristic zero. The key construction: if G is an algebraic group then it has a tangent space at the origin. The details are standard but a bit messy if you've not seen them before. Here's the idea. We have $G = \mathrm{Spec}(A)$ with A a finitely-generated reduced K-algebra. The K-point e of G gives a map $A \to K$ with kernel $\mathfrak{m}$ and $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over K of dimension $\dim(G)$. The dual of this vector space is the tangent space to G at e. A K-derivation of A is a K-linear map $D : A \to A$ with $D(ab) = D(a)b + aD(b)$. It's a pleasant exercise to check that if D and E are derivations then DE may not be, but $DE - ED$ is. The space of derivations is typically huge (it's of course $\mathrm{Hom}_A(\Omega^1_{A/K}, A)$). But we can cut it down a lot by considering derivations which commute with the G-action. Let's assume K is algebraically closed; then consider only the derivations that commute with the group action. This space is finite-dimensional and there's a natural way of identifying it with the dual of $\mathfrak{m}/\mathfrak{m}^2$, so, at least if K is algebraically closed, I've given you the construction of the Lie algebra associated to a Lie group. In the general case one can just work over the algebraic closure and then descend again but in fact we'll almost always be over $\mathbf{C}$ so I won't check the details.

One can check the Lie algebra of $SL(n)$ is $\mathfrak{sl}_n$ and so on.

lecture 6.

If K is a field and G is an affine algebraic group over K (reminder: this can be thought of as a closed subgroup of GL_n defined by polynomial equations, and the standard examples are GL_n, SL_n, Sp_{2n}, O_n). Such a thing has a tangent space at the origin which is a finite-dimensional K-vector space with a natural Lie algebra structure. One good way of seeing the Lie algebra structure is just thinking about G as a subgroup of GL_n and just believe me that the Lie algebra of GL_n is $\mathfrak{gl}_n$ and then do infinitesimal calculations to work out the subalgebra of $\mathfrak{gl}_n$ corresponding to the subgroup G. For example $\mathrm{O}(n)$ was $gg^t = 1$ and if $g = 1+\epsilon$ then we get $\epsilon+\epsilon^t = 0$ which is just the definition of $\mathfrak{so}_n$. This example actually shows an annoying thing: $\mathrm{SO}(n)$ has index 2 in $\mathrm{O}(n)$ and they have the same Lie algebra; what's going on is that $\mathrm{SO}(n)$ is the connected component of $\mathrm{O}(n)$. In general clearly $\mathfrak{g}$ doesn't determine G because G can be replaced by $G \times F$ with F finite.

So perhaps for connected affine algebraic groups, the Lie algebra determines the Lie group? Nope: $\mathbf{G}_m$ and $\mathbf{G}_a$ (both abelian algebraic groups so both have abelian 1-dimensional Lie algebras). So perhaps for connected reductive algebraic groups? Well, first I have to tell you what a reductive algebraic group is.

If G is an affine algebraic group and H is a closed subgroup then there is a natural quotient set G/H which inherits the structure of a variety but this variety may not be affine. For example GL_2 acts on $\mathbf{P}^1$. But if H is a closed normal subgroup then there is a quotient and the quotient is an affine algebraic group again. The definitions of solvable and nilpotent are just the same: if G is an affine algebraic group and H, K are closed subgroups with one of them connected then the commutator subgroup (the subgroup generated by $hkh^{-1}k^{-1}$) of H and K is again connected and closed and it's hence an algebraic group.

If G is an affine algebraic group then here's a plethora of definitions:

(i) solvable and nilpotent: you can guess. Same definition as for Lie algebras.

Let G be connected and defined over an algebraically closed field.

(ii) the *radical* $R(G)$ of G is the maximal connected solvable normal subgroup of G. It's a theorem that G has a radical (section 11.21 of Borel lin alg gps). The *unipotent radical* $R_u(G)$ of G is the maximal connected unipotent normal subgroup of G. This exists too, it's the "unipotent part" of $R(G)$.

(iii) G is *semisimple* if $R(G) = 1$ and reductive if $R_u(G) = 1$. Note that if G is an arbitrary affine algebraic group then $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive.

(iv) A *Borel subgroup* of G is a maximal connected solvable subgroup of G. These obviously exist. They are all conjugate (Theorem in section 11.1 of Borel's book on lin alg groups).

(iv) A *Parabolic subgroup* of G is a closed subgroup containing a Borel.

(v) A *Levi* subgroup of G is a connected subgroup L of G such that G is the semidirect product of L and $R_u(G)$. (these do not always exist; they exist in char 0 or if G is a parabolic subgroup of a reductive group).

(vi) A *torus* is a group isomorphic to $\mathbf{G}_m^n$ for some n. If G is any affine algebraic group then a *maximal torus* is a maximal closed connected subgroup isomorphic to a torus.

It's funny, because in the Lie Algebra case we have seen Borels before but the diagonal matrices in $\mathfrak{gl}_n$ we called a Cartan subalgebra. There is a Cartan subgroup but surprisingly it's not a maximal torus, it's a big bigger.

(vii) A *Cartan subgroup* is the centralizer of a maximal torus.

Examples: GL_n is reductive. The upper triangular matrices are a Borel. The radical of GL_n is its centre, the scalar matrices. An example of a parabolic would be (draw a staircase)[7]. The unipotent radical of the parabolic is (picture) 7
and the associated Levi is (picture). The diagonal matrices give a maximal torus, and they are their own centralizer. In general in a reductive group the centralizer of a maximal torus is itself.

In the theory of Lie algebras we might choose a cartan subalgebra and then a basis of the corresponding root system. In the theory of affine algebraic groups we start with a reductive group, choose a Borel and a maximal torus, and the Lie algebra of the maximal torus gives us a root system and the choice of Borel gives us a notion of positivity and a basis for the root system. Example of SL_3.

Now we know what a reductive group is, we can ask whether it's true that if you know the Lie algebra of a reductive group can you reconstruct the group? And it's still not true because $\mathrm{SL}_2(\mathbf{C})$ and $\mathrm{PSL}_2(\mathbf{C})$ have the same tangent space, they're both connected, one is a double cover of the other in fact, and the map on tangent spaces is an isomorphism. Furthermore you can see from this example that there are finite-dimensional representations of the Lie algebra that might not extend to finite-dimensional representations of the algebraic group: for example the natural 2-dimensional representation of $\mathfrak{sl}_2(\mathbf{C})$ is, unsurprisingly, the representation coming from the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbf{C})$. On the other hand, there is not an obvious group homomorphism $\mathrm{PSL}_2(\mathbf{C}) \to \mathrm{SL}_2(\mathbf{C})$ that I can see and in fact a theorem I tell you later will tell you all the representations of $\mathrm{PSL}_2(\mathbf{C})$ and indeed there isn't one giving rise to

[7] drawing

the standard 2-dimensional representation of $\mathfrak{sl}_2(\mathbf{C})$.

So here's how it works. Firstly I'll tell you the classification theorem for reductive groups over $\mathbf{C}$. Then I'll tell you how the representation theory of semisimple groups works.

Reminder: if V is a real vector space then a reduced root system is $R \subset V$ finite spanning V, not containing zero, and such that for any $\alpha \in R$ there is $\alpha^* \in V^*$ with $\alpha^*(\alpha) = 2$ and the function s_α on V defined by $s_\alpha(v) = v - \alpha^*(v)\alpha$ sends R to R, and such that if $\alpha, \beta \in R$ then $s_\alpha(\beta) - \beta$ is an integer multiple of α, and the only real multiples of α in R are $\pm\alpha$. These things correspond to semi-simple Lie algebras. To extend to reductive Lie algebras is easy: just make V bigger and remove the part of the definition saying that R must span V. A Cartan subalgebra for a reductive Lie algebra is just a Cartan subalgebra for the semi-simple bit, plus the centre, and the same construction gives a bijection between reduced root systems where we remove the spanning condition, and reductive Lie algebras.

It turns out that if we define $R^* \subset V^*$ to be the α^*s then R^* is a reduced root system in V^*. I should have said this earlier. So there's some kind of duality going on. The duality is almost trivial: the dual of B_n is C_n but all the others are self-dual.

Back to the point: how do I tell $\mathrm{SL}_2(\mathbf{C})$ from $\mathrm{PSL}_2(\mathbf{C})$? Here's the piece of linear algebra data. a *root datum* is a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$ where X and $X^\vee$ are finitely-generated free abelian groups and we are given a perfect pairing $X \times X^\vee \to \mathbf{Z}$ denoted $\langle , \rangle$, and Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$, and we are also given a bijection $\Phi \to \Phi^\vee$ denoted $\alpha \mapsto \alpha^\vee$. We also need two axioms: the first is that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$, and the second is the following: if $\alpha \in \Phi$ then define $s_\alpha : X \to X$ by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ and similarly $s_{\alpha^\vee} : X^\vee \to X^\vee$ is defined by $s_{\alpha^\vee}(u) = u - \langle \alpha, u \rangle \alpha^\vee$. and the second axiom is that for all $\alpha \in \Phi$ we have $s_\alpha(\Phi) = \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$. You can guess the dual of a root datum: just switch X and $X^\vee$ and Φ and $\Phi^\vee$. Given a root datum, let V be the subspace of $X \otimes \mathbf{R}$ generated by Φ: then (V, Φ) is a root system and $X \otimes \mathbf{R}, \Phi$ is an "extended root system". We say that a root datum is *semisimple* if Φ spans $X \otimes \mathbf{R}$.

Example: $X = \mathbf{Z} = X^\vee$, $\langle a, b \rangle = ab$, $\Phi = \{\pm 1\}$ and $\Phi^\vee = \{\pm 2\}$. This datum and its dual have the same associated root system. Say that a root datum is *reduced* if the associated root system is reduced.

The key construction: if G is an affine alg group and we choose a maximal torus then X is the group of characters of T, and $X^\vee$ is the group of 1-parameter multiplicative subgroups of T, that is, maps $\mathbf{G}_m \to T$, and of course Φ is the root system of G. It's a bit more delicate to define $\Phi^\vee$ but it's not too bad: if $\alpha \in \Phi$ then $\alpha : T \to \mathbf{G}_m$; let T_α denote the kernel, a subtorus of T of codimension 1. Take its centralizer; this is a connected reductive group with maximal torus T and if you quotient out by the centre then the resulting group G_α is semisimple of rank 1 and hence, by an explicit calculation, is isomorphic to SL_2 or PSL_2, so there's a unique homomorphism $\alpha^\vee : \mathrm{GL}_1 \to G_\alpha$.

Theorem 17. *Let K be an alg closed field of char 0. Given a root datum with* *reduced root system there exists reductive G and T giving rise to it; unique up to isomorphism.*

If G is semisimple then we say G is *adjoint* if $X = \langle \Phi \rangle_{\mathbf{Z}}$ and *simply-connected* if $X = \{v \in X \otimes \mathbf{R} :< x, \alpha^\vee >\in \mathbf{Z}\}$.

Need to do example of SL_2 and PSL_2.

Lecture 7

4 Infinite-dimensional representations of real and complex groups.

What people who do the Langlands program are interested in, is not primarily finite-dimensional representations, but certain infinite-dimensional ones. So let G be a connected reductive algebraic group over $k = \mathbf{R}$ or $\mathbf{C}$, and let's give a precise definition. A *representation* of $G(k)$ on a complex Hilbert space V is a homomorphism ρ from G to the group of continuous linear maps $V \to V$ with continuous inverses and such that the resulting map $G \times V \to V$ is continuous. We have the usual obvious definitions: an invariant subspace is the usual thing; V is *irreducible* if $V \neq 0$ and there are no closed invariant subspaces other than 0 or V. Say that a representation is *unitary* if for all $g \in G(k)$ the map $\rho(g) : V \to V$ is unitary (that is, satisfies $\rho(g)\rho(g)^* = \rho(g)^*\rho(g) = 1$). The advantage of unitary representations is that the complement of a closed invariant subspace is also closed and invariant. Two unitary representations on V and V' are said to be *unitarily equivalent* if there is a unitary isomorphism (that is, continuous map with continuous inverse) $V \to V'$ intertwining the representations.

One simple example: $\mathrm{SL}_2(\mathbf{R})$ acts on $L^2(\mathbf{R}^2)$ and $(\rho(g)f)(x) = f(g^{-1}x)$. This is easily checked to be unitary. Another natural representation is: choose a left Haar measure on G and let G act on $L^2(G)$.

Recall that if S is an open subset of $\mathbf{R}^n$ (for example a neighbourhood of the identity in G) and f is a function from S to a topological real vector space V, then f is said to be differentiable at $s_0 \in S$ if there is a linear map $f'(s_0) : \mathbf{R}^n \to V$ such that $\lim_{s \to s_0} \frac{f(s)-f(s_0)-f'(s_0)(s-s_0)}{|s-s_0|} = 0$ where $|.|$ is any norm on $\mathbf{R}^n$. If f is differentiable at all $s \in S$ then the map $s \mapsto f'(s)$ is a map from S to $\mathrm{Hom}(R^n, E)$ which is also a topological vector space, and we can ask whether this is differentiable. We say f is C^∞ if it's differentiable as many times as you like. If ρ is a representation of G on a complex Hilbert space then we say $v \in V$ is a C^∞ vector for ρ if the map $G \to V$ defined by $g \mapsto \rho(g)v$ is C^∞. One easily checks that this is a complex subspace[8]. It's not closed though, 8
in fact just the opposite—it's dense (Theorem 3.15 of Knapp).

If $v \in C^\infty(V)$ then define $f : \mathfrak{g} \to V$ by $f(X) = \rho(\exp(X))v$; this is C^∞; define $\phi(X)v = f'(0)X$. That is,

$$\phi(X)v = \lim_{t \to 0} \rho(\exp(tX))v - vt.$$

[8] check

If you now carefully unravel the definitions then you see that $\phi(X)$ sends $C^\infty(V)$ to itself, and that $\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$ (see Proposition 3.9 of Knapp). Hence the universal enveloping algebra acts on $C^\infty(V)$ as does it complexification. One checks that the centre of the universal enveloping algebra commutes with the G-action (corollary 3.12 of Knapp).

Now let G be a connected reductive affine algebraic group. Then $G(k)$ has a maximal compact subgroup; call it K. Rather than prove this I'll just give examples. Note that K might not be the k-points of a closed subgroup of G, it's just an abstract group (it will be a real Lie group though).

$k = \mathbf{R}$: $\mathrm{GL}_n(\mathbf{R})$ contains $O(n)$; $\mathrm{SL}_n(\mathbf{R})$ contains $\mathrm{SO}(n)$; $\mathrm{Sp}_{2n}(\mathbf{R})$ contains a group isomorphic to $U(n)$; $\mathrm{GL}_n(\mathbf{C})$ contains $U(n)$; $\mathrm{SO}_n(\mathbf{C})$ contains $\mathrm{SO}(n)$ (the matrices in $\mathrm{SO}_n(\mathbf{C})$ with real entries) and so on.

Now say G is linear connected reductive. If π is a representation of G on a Hilbert space V and if $v \in V$ then we say that v is K-finite if $\pi(K)v$ is contained in a finite-dimensional vector space. If K acts by unitary operators then we're in very good shape—the Peter-Weil theorem (Theorem 1.12 of Knapp) tells us that every irreducible unitary representation of K is finite-dimensional and hence every unitary representation is an orthogonal sum of finite-dimensional irreducible invariant subspaces (that is, the closure of the abstract direct sum). If we choose an irreducible unitary representation τ of K then denote by V_τ the space spanned by subspaces of V isomorphic to τ. One checks that the K-finite vectors in V is just the direct sum of the V_τ as τ runs through the irreducible unitary representations of K in this case. We say that τ is a *type* for K if $V_\tau \neq 0$. I will attempt to give an explicit example of this if I can work one out.

Theorem 18. *If G is linear connected reductive and π is irreducible and unitary then each V_τ is finite-dimensional; in fact* $\dim(V_\tau) \leq \dim(\tau)^2$.

Proof. Knapp Theorem 8.1. □

Definition. A representation of a linear connected reductive group G on a Hilbert space V is *admissible* if $\pi(K)$ acts as unitary operators and if each irreducible unitary representation τ of K occurs only finitely often in $\pi|K$.

So the previous theorem says that irreducible unitary representations are admissible. I suspect that historically people studied unitary representations and I suspect that it was Langlands who might have introduced this notation?

Theorem 19. *Let V be an admissible repn of linear connected reductive G. Then the K-finite vectors in V are all C^∞, and they might not be G-stable, but they are $\mathfrak{g}$-stable.*

Proof. Proposition 8.5 of Knapp. □

Note that the reason they're not G-stable is that if v is K-finite then gv is gKg^{-1}-finite and gKg^{-1} might not be commensurable with K. This is one big big difference between the archimedean and non-archimedean cases.

We say that two admissible representations of G are *infinitesimally equivalent* if the associated representations of $\mathfrak{g}$ on the K-finite vectors are algebraically equivalent. Infinitesimally equivalent does not imply isomorphic, it's weaker. But infinitesimally equivalent irreducible unitary representations are indeed unitarily equivalent, so Knapp says.

Langlands classified irreducible admissible representations of G up to equivalence and perhaps next week I will attempt to state Langlands' theorem.

Lemma 20. *If π is an admissible representation of G on V and V_0 is the K-finite vectors then $\pi(G)$ has no closed invariant subspaces iff $\pi(\mathfrak{g})$ has no non-trivial algebraic invariant subspaces on V_0.*

Proof. Corollary 8.11 of Knapp. □

If this happens we say that π is irreducible admissible. In this case, the centre of the complexified universal enveloping algebra acts as a scalar on the K-finite vectors of π (Corollary 8.14 of Knapp).

We've seen that from an admissible representation, the K-finite vectors admit a representation of $\mathfrak{g}$ and K. Here's a formal definition.

Let K a be a compact subgroup of $G(k)$, G any affine algebraic group. A $(\mathfrak{g}, K)$*-module* is a complex vector space V equipped with an action of $\mathfrak{g}_{\mathbf{C}}$ and K such that the K-representation is a (possibly infinite) algebraic direct sum of finite-dimensional representations of K (that's equivalent to every vector being K-finite) and the actions are compatible in the sense that if $X \in \mathfrak{g}$ is in the Lie algebra of K then for all $v \in V$ we have Xv (the $\mathfrak{g}$-action) is the derivative with respect to t of $\exp(tX)v$ at $t = 0$.

Oh, we need one more thing: if $k \in K$ and $X \in \mathfrak{g}_{\mathbf{C}}$ and $v \in V$ then $kXv = ((\mathrm{ad}\, k)(X))(kv)$ (here k acts on G and hence on $\mathfrak{g}$ by conjugation). A $(\mathfrak{g}, K)$-module is *admissible* if for all representations τ of K the number of times τ shows up in V is finite. A submodule is the obvious thing.

What I want to do is to work out an example: I will calculate all irreducible $(\mathfrak{gl}_2, O(2))$-modules; they will all turn out to be admissible. I'm not sure in what generality irreducible implies admissible.

Lecture 8

Recall: if G is a connected reductive affine algebraic group over $k = \mathbf{R}$ or $\mathbf{C}$ and if we choose a maximal compact subgroup K (we might have to choose one in the "correct" conjugacy class, I don't really know what's going on, but I gave a list last time) then a representation of $G(k)$ on a Hilbert space V is *admissible* if $\pi(K)$ acts via unitary operators and if each irreducible unitary representation of K only shows up finitely often. Example: any irreducible unitary representation. But there are non-unitary irreducible representations which are admissible, I think that for example induced representations might be examples of this. Let V be admissible and let V_0 denote the K-finite vectors in V. All of these vectors are C^∞ and hence $\mathfrak{g}$ (and also $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$) acts on V_0, as does K, and furthermore the actions of $\mathfrak{g}_{\mathbf{C}}$ and K are compatible, in the following sense: firstly, if $X \in \mathfrak{g}$ satisfies $\exp(X) \in K$ then the two definitions of the action of X (one via thinking it of as in $\mathfrak{g}$, the other via differentiating the K-action) are the same, and secondly the action of kX is the same as $(kXk^{-1})k$, where kXk^{-1} has a meaning in the Lie algebra, as k acts on G and hence on $\mathfrak{g}$ by conjugation.

Before I start let me give you a few definitions that I won't use again. Let V be an irreducible admissible representation of $G(k)$ with G connected linear reductive; let V_0 be the K-finite vectors. If we choose $v, w \in V$ then we get a function $G \to \mathbf{C}$ defined by $g \mapsto (gv, w)$ and this is called a *matrix coefficient.* If $v, w \in V_0$ then it's called a *K-finite matrix coefficient.* If all K-finite matrix coefficients are in $L^2(G)$ then we say that V is *discrete series*. In this case V is infinitesimally equivalent to the action of G on an irreducible closed subspace of $L^2(G)$ (Theorem 8.51 of Knapp). Note that if V is irreducible and unitary then it's enough to check that one non-zero K-finite matrix coefficient is in $L^2(G)$.

If the K-finite matrix coefficients are in $L^{2+\epsilon}(G)$ for all $\epsilon > 0$ then we say that V is *irreducible tempered.* Such a representation is infinitesimally equivalent with a unitary representation. See Theorem 8.53 of Knapp for other facts about irreducible tempered representations. Some induced representations are tempered, and others aren't.

Because I only have 1 lecture left I want to write down all the irreducible admissible $(\mathfrak{g}, K)$-modules in the case $G = \mathrm{GL}_2$ over $\mathbf{R}$, so $\mathfrak{g} = \mathfrak{gl}_2(\mathbf{R})$ and $K = O(2)$. Note that K isn't connected. Let's define $K_0 = \mathrm{SO}(2)$. Say V is an irreducible $(\mathfrak{g}, K_0)$-module. Because K_0 is isomorphic to the circle, via $e^{i\theta} = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$, we see that $V = \oplus V_n$ where $V_n = \{v \in V : e^{i\theta}.v = e^{ni\theta}v\}$. It would be nice if the eigenspaces of the action of K_0 on $\mathfrak{g}_{\mathbf{C}}$ were "easy" matrices (lots of zeroes), but unfortunately they're not. Set $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$. Then $\gamma^{-1}\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)\gamma = \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ which does act nicely on our favourite basis $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and so on. So let's choose a basis $z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $e = \gamma\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\gamma^{-1}$ and $f = \gamma\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\gamma^{-1}$ and $h = \gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\gamma^{-1}$ for $\mathfrak{g}$. Recall $[h, e] = 2e$ and $[h, f] = -2f$ and $[e, f] = h$ and everything commutes with z. We also now have that if $k = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$ then $kek^{-1} = e^{2i\theta}e$, $kfk^{-1} = e^{-2i\theta}f$ and $khk^{-1} = h$. Note also that $\exp(i\theta h)$ is just $\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in K$ and using the definition of a $(\mathfrak{g}, K)$-module gives us that ih should act via the derivative at zero with respect to θ of this, which on V_n is in, so h acts as multiplication by n on V_n.

Let H, E, F, Z denote the corresponding elements of the universal enveloping algebra. Set $\Omega = (H - 1)^2 + 4EF$. Now in $U(\mathfrak{g})$ we have $EF - FE = H$ so this is also $(H + 1)^2 + 4FE$. One checks that Ω is actually in the centre of $U(\mathfrak{g})$; this is easy, we know $U(\mathfrak{g})$ is generated by Z, E, F, H and e.g. $\Omega E = H^2E - 2HE + E + 4EFE = H(HE - 2E) + E + 4EFE$ and $HE - EH = 2E$ so this is $HEH + E + 4EFE = (2E + EH)H + E + 4EFE = E(H^2 + 2H + 1 + 4FE) = E\Omega$ and so on. In fact, although we don't need it, the centre of $U(\mathfrak{g}_{\mathbf{C}})$ is just $\mathbf{C}[\Omega, Z]$. Because V is irreducible, Ω must act by a scalar, call it ω, and Z must act by a scalar, call it z (abuse of notation but this is OK).

Now if $v \in V_n$ then $Ev \in V_{n+2}$ because $kEv = (kEk^{-1})kv$, and similarly $Fv \in V_{n-2}$. Also $EF = \frac{1}{4}(\Omega - (H - 1)^2)$ and hence $EFv = \frac{1}{4}(\omega - (n - 1)^2)v$ if $v \in V_n$ and similarly $FEv = \frac{1}{4}(\omega - (n + 1)^2)v$. In particular we see that if $\omega - (n - 1)^2 \neq 0$ and $0 \neq v \in V_n$ then $0 \neq Fv$ either.

Now *if* there exists $0 \neq v \in V_n$ such that $Fv = 0$ then $\oplus_{r \geq 0} E^r \mathbf{C}v$ is stable under E, F, H, Z and K_0, so it's V and note that this forces $\omega = (n - 1)^2$.

Similarly if there exists $0 \neq v \in V_n$ with $Ev = 0$ then $V = \oplus_{r\geq 0} F^r \mathbf{C} v$ and $\omega = (n+1)^2$. Conversely, if $\omega = (n-1)^2$ and $V_n \neq 0$ then $FV_n = 0$, and if $\omega = (n+1)^2$ and $V_n \neq 0$ then $EV_n = 0$, and in either case we now know V. **I now can't find a page of my notes so I don't know what happens next unless I work it all out again.**

Now say V is an irreducible $(\mathfrak{gl}_2, \mathrm{O}_2)$-module. It's either irreducible or reducible as a $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module. If it's irreducible then either (case A1) $\omega = m^2$ with $m \in \mathbf{Z}_{\geq 0}$ and either $V = \oplus_{n \in \mathbf{Z}, n \equiv m \bmod 2} \mathbf{C} v_n$ or $m > 0$ and $V = \oplus_{n=1-m}^{m-1} \mathbf{C} v_n$ and there are exactly two ways in which c can act. Or (case A2) $\omega \notin \mathbf{Z}^2$. Then $V = \oplus_{n \in \mathbf{Z}, n \equiv t \bmod 2} \mathbf{C} v_n$ and V is determined by z, ω, $t \bmod 2$, and a choice of a c-action (two choices again). Or (case B) V is reducible as an $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module and if $0 \subset W \subset V$ is a submodule then $V = W \oplus cW$ and we are forced to have $\omega = m^2$ with $m \in \mathbf{Z}_{\geq 0}$ and $V = \oplus_{n \leq -m-1, n \geq m+1, n \equiv m+1 \bmod 2} \mathbf{C} v_n$.

Jacquet-Langlands' description of these modules.

If B is a Borel in a connected linear algebraic group G, and V is a (say, unitary Hilbert space) representation of $B(\mathbf{R})$, then it would be nice to know how to induce it up to a representation of G but there are apparently some minor technical measure-theory problems with this induction and I won't explain how to do it properly (you look at continuous functions from G to V which transform in a certain way and then complete in the L^2 norm, so it seems). On the other hand, induction for $(\mathfrak{g}, K)$-modules seems to be much easier—I imagine this is the same as taking the usual induction and then taking the K-finite vectors. I haven't found a reference for induction on $(\mathfrak{g}, K)$-modules but here's something concrete which must be an example.

Recall from Lemma 1 that the only continuous group homomorphisms $\mathbf{R}^\times \to \mathbf{C}^\times$ are of the form $\mu(x) = |x|^s (x/|x|)^N$ with $s \in \mathbf{C}$ and $N \in \{0, 1\}$. Let mu_1, μ_2 be two such characters. Consider the pair as a character of the upper triangular matrices in $\mathrm{GL}_2(\mathbf{R})$. Now induce up: set

$$\begin{aligned} B(\mu_1, \mu_2) = &\{ f : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{C} : \\ &\quad f\left(\left(\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right)\right) g\right) = \mu_1(a)\mu_2(b)|a/b|^{1/2} f(g) \text{ and } f \text{ is } \mathrm{SO}_2(\mathbf{R})\text{-finite} \}. \end{aligned}$$

The finiteness statement is that if f satisfies the equation above then for all $k \in \mathrm{SO}_2(\mathbf{R})$ we see that the function $f_k : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{C}$ defined by $f_k(g) = f(gk)$ also satisfies the equation above, but we want the vector space generated by the f_k as k runs through K to be finite-dimensional.

Define s and N by $\mu_1 \mu_2^{-1} = |t|^s (t/|t|)^N$. Now one checks that if $n \in \mathbf{Z}$ with $n \equiv N \bmod 2$ then the function ϕ_n defined by $\phi_n(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)) = \mu_1(a)\mu_2(b)|a/b|^{1/2} e^{in\theta}$ is in $B(\mu_1, \mu_2)$ and in fact these functions form a basis of $B(\mu_1, \mu_2)$. Note that $B(\mu_1, \mu_2)$ is an admissible $(\mathfrak{gl}_2, \mathrm{O}_2)$-module. Moreover by our classification of irreducible ones we can just read off when these things are irreducible. If $s - m$ isn't an odd integer, or if it is but $s = 0$, then $B(\mu_1, \mu_2)$ is irreducible. If however $s - N$ is an odd integer and $s \neq 0$ then there are two cases. Either $s > 0$ in which case one checks that $\sigma(\mu_1, \mu_2) :=$

$\oplus_{n>=s+1,n<=-s-1}\mathbf{C}\phi_n$ is an irreducible sub and the quotient $\pi(\mu_1,\mu_2)$ is finite-dimensional and irreducible. If on the other hand $s<0$ then $1+s\leq n\leq -1-s$ gives a finite-dimensional irreducible submodule, call it $\pi(\mu_1, mu_2)$, and the quotient $\sigma(\mu_1,\mu_2)$ is irreducible. One now checks without too much trouble that every irreducible admissible $(\mathfrak{gl}_2(\mathbf{R}),\mathrm{O}_2)$-module is either a π or a σ, and that the only isomorphisms between them are $\pi(\mu,\nu)=\pi(\nu,\mu)$ and $\sigma(\mu,\nu)=\sigma(\nu,\mu)=\sigma(\mu\eta,\nu\eta)=\sigma(\nu\eta,\mu\eta)$ where $\eta(t)=t/|t|$. The dictionary is that if $\mu_i(t)=|t|^{s_i}(t/|t|)^{N_i}$ then $z=s_1+s_2$, $\omega=(s_1-s_2)^2$ on $B(\mu_1,\mu_2)$.

Note that there is something funny going on here: the picture is much simpler than in the p-adic case. In the p-adic case we could induce up characters of the Borel, and we got principal series and special representations, but there were also strange supercuspidal representations which didn't arise as a subquotient of principal series. In the case of $\mathrm{GL}_2(\mathbf{R})$ this doesn't happen.

Now Jacquet and Langlands manage to associate an L-function to the σs and πs but the constructions are quite messy, involving re-interpreting the representations as certain spaces of functions on $\mathrm{GL}_2(\mathbf{R})$ a la Whittaker model.

The $\pi(\mu,\nu)$ correspond to the reducible Galois representation corresponding to $\mu\oplus\nu$. If $\mu(t)=|t|^z$ then $L(\mu,s)=\pi^{-(s+z)/2}\Gamma((s+z)/2)$ and if $\mu(t)=|t|^z(t/|t|)$ then $L(\mu,s)=\pi^{-(s+z+1)/2}\Gamma((s+z+1)/2)$. Finally the $\sigma(\mu,\nu)$ case: if $\sigma(\mu,\nu)$ is defined then after re-ordering we have $\mu/\nu(t)=t^s(t/|t|)$ with $s\in\mathbf{Z}_{>0}$. If ν is trivial and I have this right then this corresponds to the 2-dimensional representation of $W_{\mathbf{R}}$ induced by the representation $z\mapsto z^{-s}$ of $W_{\mathbf{C}}=\mathbf{C}^\times$. The general case follows by twisting, and the fact that $\sigma(\mu,\nu)=\sigma(\mu\eta,\nu\eta)$ corresponds to the fact that twisting the induced representation by η doesn't change it.

I think that it must be the case that all of the irreducible $(\mathfrak{gl}_2,\mathrm{O}_2)$-modules we wrote down do show up as the K-finite vectors in a Hilbert space representation of $\mathrm{GL}_2(\mathbf{R})$; if we believe this then we've proved

Theorem 21. *There's a natural bijection between the 2-dimensional representations of $W_{\mathbf{R}}$ and infinitesimal equivalence classes of admissible representations of $\mathrm{GL}_2(\mathbf{R})$.*

To get ones hands on what's going on in the general case, one has to understand induction properly. Basically one should stick to inducing twists of unitary representations, so it seems. And here there's a trick: there is a unitary discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ corresponding to the σ's. One can write it down explicltly: given an integer $s\geq 1$ one defines D_s^+ to be the representation of $\mathrm{SL}_2(\mathbf{R})$ acting on the analytic f on the upper half plane and satisfying $||f||=\int\int|f(z)|^2y^{s-1}dxdy<\infty$, with the action being $(gf)(z)=(bz+d)^{-(s+1)}f((az+c)/(bz+d))$ and then one induces this up to $\mathrm{SL}_2^{\pm}(\mathbf{R})$ and gets a representation of $\mathrm{GL}_2(\mathbf{R})$ in what I presume is the usual way. Now one thinks about the πs as coming from induced representations from the Borel, but the σs as coming from twists of these discrete series. I don't really understand why. It seems to me that any irreducible admissible representation of GL_n should be infinitesimally equivalent to something induced from a Borel but the way people explain it is to induce from a slightly bigger group. Somehow it fits best into Langlands' framework in this setting. Langlands proved that inducing up twists of tempered unitary representations from parabolics gave all irreducible admissible representations of an arbitrary reductive group G up to infinitesimal equivalence. This is known as Langlands' classification of irreducible admissible representations. To deal with the tempered representations though, one runs into the theory of L-packets and I'm not qualified to talk about this so I'll stop.