Notes on Siegel modular forms.

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1 Definitions and the basics.

this document was written in about 2004 when supervising Joshua Harris for his 4th year undergraduate project at Imperial.

For simplicity I will stick to "full level structure", indicating why I've made this choice a bit later on. Let $n \geq 1$ be an integer (the case $n = 0$ is sometimes useful but we shall refrain from doing empty-set theory and just give ad-hoc definitions). If A and B are $2n$ by $2n$ matrices then by the notation $A[B]$, I mean $(B^t)AB$. Let I denote the n by n identity matrix and let J denote the $2n$ by $2n$ matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

For a ring R we define $\mathrm{GSp}(2n, R)$ to be the invertible $2n$ by $2n$ matrices M such that $J[M] = \nu(M)J$, for $\nu(M) \in R^\times$, and $\mathrm{Sp}(2n, R)$ to be the kernel of ν.

We define Siegel upper half space $\mathbf{H}_n^+$ to be the set of n by n complex matrices $\Omega = X + iY$ such that $\Omega^t = \Omega$ and such that Y is positive definite. This space has dimension $n(n+1)/2$ and useful coordinates are the (i, j)th entries of Ω for $i \leq j$. Recall that $\mathrm{Sp}(2n, \mathbf{R})$ acts on $\mathbf{H}_n^+$ by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\Omega = (A\Omega + B)(C\Omega + D)^{-1}$. Here of course A, B, C, D denote n by n matrices. The proof that this is well-defined and an action is in my notes on Shimura varieties. One thing I didn't mention in those notes was the following pleasant exercise: the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathrm{Sp}(2n, R)$ with $C = 0$ are precisely the matrices of the form $\left(\begin{smallmatrix} U & SU^{-t} \\ 0 & U^{-t} \end{smallmatrix}\right)$ with $U \in \mathrm{GL}_n(R)$, S symmetric and U^{-t} meaning the inverse transpose of U. These matrices are called "integral modular substitutions" by some people. Note that the matrices can also be written $\left(\begin{smallmatrix} U & UT \\ 0 & U^{-t} \end{smallmatrix}\right)$ with T symmetric, notation which is easier to remember but whose action on $\mathbf{H}_n^+$ is messier.

Now let k be an integer. We define a Siegel modular form of genus n and weight k to be a holomorphic function $f : \mathbf{H}_n^+ \to \mathbf{C}$ such that for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathrm{Sp}(2n, \mathbf{Z})$ we have $f(\gamma\Omega) = \det(C\Omega + D)^k f(\Omega)$, plus some boundedness conditions. These conditions are automatically satisfied if $n \geq 2$ so we won't mention them. If $n = 1$ then we want that f is bounded on a fundamental domain. If $n = 0$ then it seems to me that the constants should be modular forms of weight k, for any k, at least if $k \geq 0$ (I read once that for $k < 0$ the space should be zero-dimensional but I'm not sure I ever care about $k < 0$).

There is a good notion of Fourier expansion for these things. One sees this as follows: applying the functional equation to the case of integral modular substitutions shows that if f is a Siegel modular form then $f(U\Omega U^t + S) = (\det U)^k f(\Omega)$ for all $U \in \mathrm{GL}_n(\mathbf{Z})$ and all integral symmetric S. In particular $(U = I)$ f is periodic with period 1 in each of its $n(n+1)/2$ variables. Now by some version of Cauchy's integral formula which works in more than one variable and which I am happy to believe exists, we see that f has a Fourier expansion in terms of the variables $q_{i,j}$, $1 \leq i \leq j \leq m$ where $q_{i,j} = e^{2\pi\sqrt{-1}z_{i,j}}$ and $z_{i,j}$ is the (i,j)th entry of Ω. We deduce the following useful computational crutch: $f = \sum_{a_{i,j}\in\mathbf{Z}} c(a_{i,j}) \prod_{i\leq j} q_{i,j}^{a_{i,j}}$, where the sum is over integers $a_{1,1}$, $a_{1,2}$ and so on up to $a_{n,n}$, and the $c(a_{i,j}) = c(a_{1,1}, a_{1,2}, \ldots)$ are complex numbers. Note that $\prod_{i\leq j} q_{i,j}^{a_{i,j}}$ is $e^{2\pi\sqrt{-1}T}$ where $T = \sum a_{i,j} z_{i,j}$ which is the trace of $A\Omega$, with A the symmetric matrix whose (i,j)th entry is $a_{i,j}/2$ if $i < j$, and $a_{i,i}$ if $i = j$. It's convenient sometimes to let $c(A)$ denote the Fourier coefficient $c(a_{i,j})$ corresponding to this matrix. If one now allows more general elements of $\mathrm{GL}_n(\mathbf{Z})$ for U, one sees that for $U \in \mathrm{GL}_n(\mathbf{Z})$ we have $c(A[U]) = \det(U)^k c(A)$. If n is odd then setting $U = -I$ we deduce that there are no non-zero forms of odd weight. If n is even this isn't true, there is for example a non-zero form of weight 35 if $n = 2$.

It is a theorem of Koecher, using the fact that $n \geq 2$, that for a Siegel modular form we have $c(A) = 0$ will vanish unless the half-integral symmetric matrix A is positive semi-definite. One can also deduce from these arguments that there are no non-zero forms of negative weight, and that the only forms of weight 0 are the constants. Other analytic arguments show that the dimension of the weight k forms is finite, and has size $O(k^{n(n+1)/2})$.

2 Eisenstein series and the Φ operator.

There is a map Φ from weight k forms of genus n to weight k forms of genus $n-1$ (recall we always have full level structure), which can simply be defined by replacing all the $q_{i,n}$ by zero. Note that if $n = 1$ then this just the map sending a modular form to its constant coefficient, which we are regarding as a "modular form of genus 0", and if $n = 0$ then we define Φ to be the zero map. Note that it's an easy check that all one has to to is replace $q_{n,n}$ by zero because symmetric positive semi-definite matrices with 0 in the bottom right hand corner will have zeroes all along the bottom row. Now because $c(A) = c(UAU^t)$ we see that the kernel of Φ is exactly the Siegel modular forms which have Fourier expansions with $c(A) = 0$ unless A is actually positive definite. A *cusp form* is something in the kernel of Φ.

As in the classical case, if $k > n+1$ and k is even then there is an Eisenstein series $E_k(\Omega)$. In fact more is true: if $0 \leq r < n$, if $k > n+r+1$ is even and if f is a genus r cusp form then there's an Eisenstein lift of f to a genus n non-cusp form. For example if $r = 0$ then $E_k(\Omega) = \sum_\gamma \det(C\Omega + D)^{-k}$ is an Eisenstein series where the sum is over all matrices $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $M\backslash\mathrm{Sp}(2n, \mathbf{Z})$, where M is the subgroup of integral modular substitutions. Here we're regarding the constants as cusp forms of weight k and genus 0. This construction is a one-sided inverse to Φ^{n-r}. In particular one deduces that if M_k are modular forms of weight k and genus n, and S_k are cusp forms, then for even $k > 2n$ we have M_k/S_k isomorphic, as a vector space, to modular forms of weight k and genus $n-1$.

There is a Petersson inner product on cusp forms, and it extends to a pairing between cusp forms and modular forms. Hence one can decompose M_k into a direct sum of S_k and N_k, the orthogonal complement. Now if $k > 2n$ is even then Φ is an isomorphism from N_k to modular forms of genus $n-1$ and weight k, and one can continue to decompose N_k.

But here's a big difference with the classical theory: if $n > 1$ then it appears that there is no general computationally useful well-known formula for the Fourier coefficients of the Eisenstein series! Weird. One can do something though, for example Skoruppa gives a practical algorithm for working out E_4 when $n = 2$, and several other things.

3 The structure theorem for $n = 2$.

I will now restrict to the case $n = 2$ because it's here that the structure theorem (analogue of $\mathbf{C}[E_4, E_6]$ in $n = 1$ case) is well-known (work of Igusa). Notation: M_k will now denote the genus 2 modular forms of weight k and S_k the cusp forms. It turns out that $M_0 = \mathbf{C}$ and $M_k = 0$ for $1 \leq k \leq 3$. Recall that there is a map Φ sending M_k to classical modular forms of weight k, and this is always surjective—we wrote down a one-sided inverse if $k > 4$ was even, surjectivity is trivial if $k > 4$ is odd, and if $k = 4$ one simply checks that M_4 is one-dimensional and that the map sends a generator to something non-zero. One can do this by explicitly computing a non-zero element of M_4—see for example Skoruppa's explicit formula in his 1992 Math Comp paper.

For our variables in the $n = 2$ case, one can write $\Omega = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ with τ, τ' in the usual complex upper half plane, and $z \in \mathbf{C}$. Now set $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$ and $q' = e^{2\pi i\tau'}$. The Fourier expansion of a genus 2 Siegel modular form F (now using n for another meaning rather than the genus) looks like

$$F = \sum_{r,n,m \in \mathbf{Z}} a(n, r, m) q^n \zeta^r (q')^m$$

with $a(n, r, m) \in \mathbf{C}$. The "correct" way to think about the triple (n, r, m) is to think of the corresponding quadratic form $nX^2 + rXY + mY^2$, or the associated matrix $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$. Andrianov uses the normalisation $\left(\begin{smallmatrix} 2n & r \\ r & 2m \end{smallmatrix}\right)$ by the way, making things nice and integral but giving lots of bogus "2"s around in the $n = 1$ case.

As mentioned above, the coefficient $a(n, r, m)$ will automatically vanish unless $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ is positive semi-definite, and coefficients will be equal if the corresponding matrices give equivalent (that is, in the same $\mathrm{GL}_2(\mathbf{Z})$ class) bilinear forms.

Igusa shows that the direct sum of all the even weight genus 2 Siegel forms is $\oplus_{k \geq 0, 2|k} M_k = \mathbf{C}[E_4, E_6, E_{10}, E_{12}]$. Well, he used two other forms (cusp forms) of weight 10 and 12 but in Klingen's book, chapter 9, he remarks that one can use Eisenstein series instead. On the other hand, Skoruppa follows Igusa and gives explicit formulae for the cusp forms of weights 10 and 12 that Igusa uses. Once one knows explicit formulae for Fourier expansions, one can of course hope to compute Hecke operators. Just before I explain this, I will remark about the explicit decomposition of the space of modular forms. Let $k > 4$ be an even integer. Then M_k is the direct sum of the space of cusp forms, and the space N_k which is isomorphic to classical modular forms of weight k (and hence N_k has a bit corresponding to classical weight k cusp forms, namely the Eisenstein series associated to that form, and also a bit corresponding to the constants, that is, the Eisenstein series coming from $n = 0$). I think that this space is deemed "uninteresting" as a Hecke module; one should be able to read off all Hecke eigenvalues from Langlands' philosophy and no doubt the explicit results are all proved in this case. On the Galois side the construction sends ρ_f to $\rho_f \oplus (\omega^{k-2} \otimes \rho_f)$, with ω the cyclotomic character.

But there is another interesting space in the cusp forms—it comes from classical cusp forms of weight $2k-2$, via Jacobi forms of weight k and index 1. On the Galois side, this corresponds to the Siegel modular forms having a 4-dimensional Galois representation which is $\omega^{k-2} \oplus \omega^{k-1} \oplus \rho_f$.

4 The Galois side for $n = 2$.

Recall that GSp_4 looks like matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with AB^t and CD^t symmetric and $AD^t - BC^t$ scalar.

Let k be a non-negative even integer. Then the space of Siegel modular forms breaks up into the direct sum of 4 subspaces, as we explain below. NB: I learnt most of this from Skoruppa's paper on computing Siegel modular forms.

1) The first is "very" Eisenstein—the 1-dimensional (or 0-dimensional if $k = 2$) space coming from GSp_0, which can also be thought of as the $n = 2$ Eisenstein series coming from the $n = 1$ Eisenstein series. The associated Galois representation is diagonal, isomorphic to $1 \oplus \omega^{k-1} \oplus \omega^{k-2} \oplus \omega^{2k-3}$. NB: for $k \leq 8$ this is everything.

2) The second is still Eisenstein, but a bit less so—it's coming from classical ($n = 1$) cusp forms via the Eisenstein series construction of the last section. The space is isomorphic to the space of classical ($n = 1$) weight k cusp forms and on the Galois side the construction sends ρ_f to $\rho_f \otimes (1 \oplus \omega^{k-2})$. Note that this construction kicks in at $k = 12$. Note also that one can think of the image as being contained in the matrices in GSp_4 which are "block diagonal" of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ with AD^t scalar. This group is abstractly isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$. Note that ρ_f is a twist of its dual, so everything here is consistent.

3) The third subspace is cuspidal, in the sense that it is composed of Siegel cusp forms. But the associated 4-dimensional Galois representations are still

reducible. The image is contained in matrices of the form

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & \lambda & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$$

with $ad - bc = \lambda\mu \neq 0$. This group is also abstractly isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$ and David Whitehouse informs me that it is also contained within a parabolic—the so-called Klingen parabolic. So I have no idea why the associated automorphic form is cuspidal. It's something to do with choosing the special representation rather than the principal series one at infinity. Perhaps the Galois representation is the wrong thing to be looking at. Actually perhaps the Galois representation isn't contained in what I wrote above—perhaps there is some kind of non-semi-simplicity or something? All of these representations come from classical ($n = 1$) weight $2k - 2$ cusp form so in particular they kick in at $k = 10$ with Kurokawa's first counterexample to the generalised Ramanujan conjecture. On the Galois side the representation is $\rho_f \oplus \omega^{k-1} \oplus \omega^{k-2}$, with the ρ_f being the GL_2 part and the λ and μ coming from the cyclotomic characters.

4) The last part is the "interesting" cuspidal part, that part which isn't explained by any smaller group. This kicks in at weight 20 with a 1-dimensional space (Kurokawa also found this form). The dimensions for $k = 20, 22, 24, 26, 28$ are $1, 1, 2, 2, 3$. Skoruppa noticed the very strange (in my opinion!) fact that the 2-dimensional spaces at weights 24 and 26 are spanned by eigenforms with rational Fourier expansions, rather than conjugate eigenforms defined over some quadratic extension with random discriminant, as in the case of classical cusp forms of weight 24.

See Skoruppa's Math Comp 1992 paper for tables of dimensions.

5 Hecke operators for $n = 2$

Let $\Gamma = \mathrm{Sp}_4(\mathbf{Z})$ and let Δ denote the 4 by 4 integral matrices in $\mathrm{GSp}_4(\mathbf{Q})$ with positive ν. If $\delta = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Delta$ then, following Panchishkin, who is following something else (in Russian), for a function $f : H_2^+ \to \mathbf{C}$, we define $f|\delta : H_2^+ \to \mathbf{C}$ by

$$(f|\delta)(\Omega) = \nu(\delta)^{2k-3} \det(C\Omega + D)^{-k} f(\delta\Omega).$$

Note that $\det(\delta) = \nu(\delta)^2$ so we could replace $\nu(\delta)^{2k-3}$ by $\det(\delta)^{k-3/2}$ if we take the positive square root. With this choice of normalisation, we should be able to follow Panchishkin's normalisations for L-functions.

We first recall some of the abstract theory of Hecke rings, specialised to the case $n = 2$. A terrific reference for this is Andrianov's "Quadratic forms and Hecke operators", sections 3.2 and 3.3 and so on. A lot of the stuff here goes through for general n of course, I have translated down to the case $n = 2$. The Hecke algebra in question is the one associated to the group Γ in the semi-group Δ consisting of the elements of $\mathrm{GSp}_4^+(\mathbf{Q})$ with integer entries, where the

$^+$ denotes positive ν. (If one is working locally at some prime p one might also want to enlarge Δ by adding the diagonal matrix with p^{-1}s on the diagonal.)

One very useful fact is that every double coset $\Gamma\delta\Gamma$ contains a unique element s of the form $\mathrm{diag}(d_1, d_2, e_1, e_2)$ with $d_1|d_2|e_2|e_1$ and $d_1e_1 = d_2e_2 = \nu(s)$.

If $n \geq 1$ is an integer, we define $T(n)$ to be the Hecke operator equal to the sum of the double cosets $[\Gamma\delta\Gamma]$, where δ runs through the matrices of the form $\mathrm{diag}(d_1, d_2, e_1, e_2)$ as above, with $\nu(\delta) = n$. Example: $T(p) = [\Gamma\,\mathrm{diag}(1, 1, p, p)\Gamma]$ and

$$T(p^2) = [\Gamma\,\mathrm{diag}(1, 1, p^2, p^2)\Gamma] + [\Gamma\,\mathrm{diag}(1, p, p^2, p)\Gamma] + [\Gamma\,\mathrm{diag}(p, p, p, p)\Gamma].$$

We also define the Hecke operator $S_p = [\Gamma\,\mathrm{diag}(p, p, p, p)\Gamma]$. Note that, by our normalisations, S_p acts as p^{2k-6} on forms of weight k.

We have already defined $T(p)$. Write $T_p = T(p)$. Define

$$T_1(p^2) = [\Gamma\,\mathrm{diag}(1, p, p^2, p)\Gamma]$$

and $T_2(p^2) = S_p$. Now Theorem 3.3.23 of Andrianov tells us that the abstract Hecke ring at p (over $\mathbf{Q}$—I don't know if there are issues if one works over $\mathbf{Z}$) is a polynomial ring on T_p, S_p and $T_1(p^2)$, and the Hecke ring of spherical functions defined using convolution and so on is just the localisation of this ring gotten by inverting S_p.

One of course needs to know much more than this—one needs to know the Satake parameters, which it seems are traditionally called x_0, x_1, x_2. The Satake isomorphism is a map from the local Hecke operator at p to the ring $\mathbf{Q}[x_0, x_1, x_2, (x_0x_1x_2)^{-1}]$; it's an injection and an isomorphism onto its image, which is the W-invariant elements of this ring. Here W is the Weyl group, isomorphic to D_8, and it is generated by the following three transpositions:

1) $x_0 \mapsto x_0x_1, x_1 \mapsto x_1^{-1}, x_2 \mapsto x_2$
2) $x_0 \mapsto x_0x_2, x_1 \mapsto x_1, x_2 \mapsto x_2^{-1}$
3) $x_0 \mapsto x_0, x_1 \mapsto x_2, x_2 \mapsto x_1$.

Note that there is an asymmetry here—x_0 is different from the rest. Now in terms of the representation theory, after staring at the definitions of the L-functions in Andrianov and Panchishkin, and after one realises that the Langlands dual of GSp_4 is probably something like $GSpin(5)$ which happens to be abstractly isomorphic to GSp_4 anyway but whose natural representation is 5-dimensional, we see that the eigenvalues of the "standard" (5d) representation should be $x_0^2x_1x_2(1, x_1, x_1^{-1}, x_2, x_2^{-1})$ and that the eigenvalues of the "spinor" (4d) representation should be $(x_0, x_0x_1, x_0x_1x_2, x_0x_2)$. Note that in general these representations have size $2n + 1$ and 2^n, so $n = 2$ is the last time one sees representations of dimension $2n$.

To explicitly see the isomorphism, one could use coset decompositions, because the Satake isomorphism is completely explicit once one has these decompositions. In fact, here is the Satake isomorphism: given a coset $(\Gamma\delta)$, by Lemma 3.3.11 of Andrianov one can choose δ of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$; here D is uniquely defined modulo left multiplication by $\mathrm{GL}_2(\mathbf{Z})$, B has the property that D^tB is symmetric of course, and furthermore is well-defined "mod D" in the sense that $B \equiv B'$ mod D iff $(B-B')D^{-1} \in \mathrm{M}_2(\mathbf{Z})$. Finally $A = \nu(\delta)D^{-t}$ of course. If δ is of this form then we can even refine D, following Lemma 3.2.7 of Andrianov, so that it is upper triangular, and then the diagonal entries are uniquely determined. We are working at p so the diagonal entries of D are $(p^{\delta_1}, p^{\delta_2})$, and $\nu(\delta) = p^{\delta}$ (with apologies for two δs, they will go away in a second). The Satake isomorphism sends $(\Gamma\delta)$ to $x_0^{\delta}(x_1/p)^{\delta_1}(x_2/p^2)^{\delta_2}$.

I feel that at the end of the day I'll have to work out these decompositions anyway, to see how they act on Fourier expansions! But nonetheless, we can avoid them for the time being. Theorem 3.3.30 of Andrianov says that if H denotes the subring of $\mathbf{Q}[x_0, x_1, x_2, (x_0x_1x_2)^{-1}]$ fixed by W then $H \cap \mathbf{Q}[x_0, x_1, x_2]$ is the polynomial ring generated over $\mathbf{Q}$ by $t := x_0(1+x_1)(1+x_2)$ (the trace of the 4-dimensional representatiion), $\rho_0 := x_0^2x_1x_2$ (the norm of the 4-dimensional representation) and $\rho_1 := x_0^2x_1x_2(x_1 + x_1^{-1} + x_2 + x_2^{-1})$, and when we invert ρ_0 we recover H. By the remark a few lines down on p158 of Andrianov, we see that S_p gets sent to $p^{-3}\rho_0$. In fact one doesn't need to invoke any big theorems here, one just notes that with the notation above, there is only one left coset $\Gamma\,\mathrm{diag}(p,p,p,p)$ and so $\delta = 2$ and $\delta_1 = \delta_2 = 1$ and that's it.

Using Lemma 3.3.32 and Lemma 3.3.34 we see that Satake sends $T(p)$ to t (see 3.3.70). Using the remarks on p164 we see that Satake sends $T(p^2)$ to a very messy polynomial in x_0, x_1 and x_2 which can be simplified to $t^2 - \rho_1 - (2+1/p)\rho_0$. Using Lemma 3.3.32 and Lemma 3.3.34 (and suppressing a lot of calculations) one sees that the image of $T_1(p^2)$ is the sum of the images of $\Pi_{1,1}^{(0)}$, $\Pi_{2,0}^{(1)}$ and $\Pi_{1,0}^{(0)}$ which one can work out explicitly: I got that $\Pi_{1,1}^{(0)}$ was sent to $\frac{1}{p}\rho_0(x_1+x_2)$, that $\Pi_{2,0}^{(1)}$ was sent to $(p^2-1)x_0^2/p^3x_1x_2$ and that $\Pi_{1,0}^{(0)}$ was sent to $x_0^2(x_1+x_2)/p$. Hence $T_1(p^2)$ is sent to $\frac{\rho_1}{p} + \frac{p^2-1}{p^3}\rho_0$. The analogous calculation would clearly be much harder if $n \geq 3$ because one would presumably then see harder matrices in GL_3. What I am saying here is that there is no explicit analogue of lemmas 3.3.32 and 3.3.34 in the GL_n case—one has Lemma 3.3.21 but I don't see any explicit evaluations of the $\pi_{\alpha\beta}$ in the GL_n case, and we are lucky that the only ones that show up when $n = 2$ can be broken back down to π_is.

As a check, we see that there is an element $q_2^2(p) = 2\rho_0 + \rho_1$ in the Hecke algebra, defined in Proposition 3.3.35, and exercise 3.3.38 one is asked to check it's $pT_1(p^2) + (p^3+p)S_p$, and this all comes out. Hooray.

Whilst we're here, we may as well compute everything, to check that it all adds up. There is an operator $T_0(p^2)$ which is really just $\mathrm{diag}(1,1,p^2,p^2)$. Using the lemmata 3.3.32 and 3.3.34 we see that $T_0(p^2)$ gets sent to the sum of $\Pi_{a,b}^{(a)}$ for $(a,b) \in \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)\}$. Six elements! Ugh. Working this all out explicitly we see that $T_0(p^2)$ is $x_0^2(\pi_{0,0} + p^2\pi_{0,1} + p^6\pi_{0,2} + (p-1)\pi_{1,0} + (p-1)\pi_{1,1} + (p^3-p^2)\pi_{2,0})$ (where I have omitted lots of troublesome symbols) and these πs are thought of in the GL(2) Hecke algebra that is talked about in section 3.2 of Andrianov. Hence $T_0(p^2) = x_0^2(1 + p^2\left(\begin{smallmatrix}1&0\\0&p^2\end{smallmatrix}\right) + p^6\left(\begin{smallmatrix}p^2&0\\0&p^2\end{smallmatrix}\right) + (p-1)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right) + p^3(p-1)\left(\begin{smallmatrix}p&0\\0&p^2\end{smallmatrix}\right) + (p^3-p^2)\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)$. Now for GL_2 the way it works is that $\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)$ goes to $(x_1+x_2)/p$ and $\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)$ goes to $(x_1x_2)/p^3$ so all that we need to know now is that in the Hecke algebra for GL_2 we have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)^2$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^2 \end{smallmatrix}\right) + (p+1)\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ so $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^2 \end{smallmatrix}\right)$ goes to $(px_1^2 + (p-1)x_1x_2 + px_2^2)/p^3$. Now writing everything down explicitly, we see that $T_0(p^2) = x_0^2(1 + (px_1^2 + (p-1)x1x2 + px_2^2)/p + x_1^2x_2^2 + (p-1)(x_1+x_2)/p + (p-1)/p(x_1+x_2)(x_1x_2) + (p-1)/p(x_1x_2)) = t^2 - (1+1/p)\rho_1 - (2+2/p)\rho_0$.

6 Summary of all that mess.

The Hecke algebra at p can be regarded as the subring of the abstract ring $\mathbf{Q}[x_0, x_1, x_2, x_0^{-1}, x_1^{-1}, x_2^{-1}]$ fixed by the Weyl group. This ring is isomorphic to $\mathbf{Q}[t, \rho_0, \rho_1, \rho_0^{-1}]$ where $t = x_0(1+x_1)(1+x_2)$, $\rho_0 = x_0^2x_1x_2$ and $\rho_1 = \rho_0(x_1 + x_1^{-1} + x_2 + x_2^{-1})$. The dictionary is as follows: $T(p) := \mathrm{diag}(1,1,p,p)$ goes to t, $T_0(p^2) := \mathrm{diag}(1,1,p^2,p^2)$ goes to $t^2 - (1+1/p)\rho_1 - (2+2/p)\rho_0$, $T_1(p^2) := \mathrm{diag}(1,p,p^2,p)$ goes to $\rho_1/p + (p^2-1)/p^3\rho_0$, $T_2(p^2) := S_p := \mathrm{diag}(p,p,p,p)$ goes to ρ_0/p^3 and $T(p^2) := T_0(p^2) + T_1(p^2) + T_2(p^2)$ (every matrix with $\nu = p^2$) goes to $t^2 - \rho_1 - (2+1/p)\rho_0$.

The characteristic polynomial of Frob_p for the canonical 4-dimensional representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ associated to a normalised form will be $(X - x_0)(X - x_0x_1)(X - x_0x_1x_2)(X - x_0x_2)$ and this comes out to be

$$X^4 - tX^3 + (\rho_1 + 2\rho_0)X^2 - t\rho_0X + \rho^2.$$

7 Explicit computations of Hecke operators.

Note that lemma 3.3.11 of Andrianov is very helpful for finding coset representatives.

I have done the following decomposition: Let Γ denote $\mathrm{Sp}_4(\mathbf{Z})$ and set $\epsilon = \mathrm{diag}(1,1,p,p)$;

Proposition 1. $\Gamma\epsilon\Gamma = \coprod \Gamma\alpha$, *where* α *runs through the following* p^3+p^2+p+1 *elements:*

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} p & 0 & 0 & 0 \\ -\alpha & 1 & 0 & d \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & d \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Here $0 \le \alpha, a, b, d \le p-1$.

Proof. Note that $\Gamma\epsilon\Gamma = \Gamma\delta\Gamma$ where $\delta = \mathrm{diag}(p,p,1,1)$, because $\pm J \in \Gamma$. Write $\Gamma' = \delta^{-1}\Gamma\delta$. Note that $\Gamma\delta\Gamma = \coprod \Gamma\alpha$ iff $\Gamma'\Gamma = \delta^{-1}\Gamma\delta\Gamma = \coprod \delta^{-1}\Gamma\alpha = \coprod \Gamma'\delta^{-1}\alpha$. Then quotienting out on the left by Γ' we see that indexing set on the right must have size equal to the size of $\Gamma\cap\Gamma'\backslash\Gamma$. The intersection $\Gamma\cap\Gamma'$ is just the matrices in Γ which mod p have a two-by-two zero matrix in the bottom left hand corner. So now we can reduce everything modulo the kernel of $\mathrm{Sp}_4(\mathbf{Z}) \to \mathrm{Sp}_4(\mathbf{Z}/p\mathbf{Z})$, remarking that by Andrianov Lemma 3.3.2 this map is surjective (in fact this somehow came out in the wash from my first proof of the proposition!) and so the size of the index set on the right is equal to the index of the parabolic $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ in $\mathrm{Sp}_4(\mathbf{Z}/p\mathbf{Z})$.

Now let's count: computing choices for e_1, f_1, e_2, f_2 shows that the size of $\mathrm{Sp}_4(\mathbf{Z}/p\mathbf{Z})$ is $(p^4-1)(p^3)(p^2-1)(p)$, and the known structure of the matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ shows that there are $(p^2-1)(p^2-p)$ choices for A and p^3 choices for B, and then D is determined. Hence the number of cosets that we're trying to compute is $(p^4(p^4-1)(p^2-1))/((p^2-1)(p^2-p)p^3) = 1+p+p^2+p^3$.

All we have to do now is to check that the $1+p+p^2+p^3$ elements above are all in $\mathrm{Sp}_4(\mathbf{Z})$ (they weren't in the first version of this proposition, but they are now!), and that the $1+p+p^2+p^3$ cosets they generate are distinct. Now $\Gamma\alpha = \Gamma\beta$ iff $\beta\alpha^{-1} \in \Gamma$ and because all the matrices are in $\mathrm{GSp}_4(\mathbf{Q})$ and all have $\nu = p$ we see that $\Gamma\alpha = \Gamma\beta$ iff $\beta\alpha^{-1}$ has integer entries. Now looking at top left and bottom right hand entries of the matrices in the list we have, we see that if α and β are on the list and $\Gamma\alpha = \Gamma\beta$ then α and β must have the same top left and bottom right hand 2 by 2 blocks. So we are reduced to checking each of the four families individually, each of which is an easy check.

□

Somehow this proposition must be embedded in Andrianov. Note that because we have grouped everything together in terms of the bottom right hand matrix, this means that we can compute the action on Fourier expansions individually on these four pieces. This is all very easy now.

Lemma 1. *If* $F = \sum_{n,r,m} a(n,r,m)q^n\zeta^r(q')^m$ *then*

$$T(p)F = \sum_{n,r,m} b(n,r,m)q^n\zeta^r(q')^m,$$

where

$$b(n,r,m) = p^{2k-3}a(n/p,r/p,m/p) + p^{k-2}a(pn,r,m/p)$$
$$+p^{k-2}\sum_{0\le\alpha<p} a\left((n+r\alpha+m\alpha^2)/p, r+2m\alpha, pm\right) + a(pn,pr,pm),$$

where $a(n,r,m) = 0$ *if* n,r,m *aren't all integers.*

Proof. Just bash it out from the double coset decomposition. Let's see what happens to the term $q^n\zeta^r(q')^m$. The first term sends this to $p^{2k-3}q^{pn}\zeta^{pr}(q')^{pm}$. The second terms send it to

$$p^{2k-3-k}\sum_d q^{p/n}\zeta^r(q')^{mp}(\zeta_p)^{nd/p}$$

which is

$$p^{k-2}q^{n/p}\zeta^r(q')^{mp}$$

if $p|n$ and 0 otherwise. The third terms are the messiest. We get

$$p^{2k-3-k}\sum_{\alpha,d} q^{np-\alpha r+\alpha^2 m/p}\zeta^{r-2\alpha m/p}(q')^{m/p}(\zeta_p)^{md}$$

and summing over d means that this vanishes if $p \nmid m$ but if $p|m$ it's

$$p^{k-2} \sum_{\alpha} q^{np-\alpha r+\alpha^2(m/p)} \zeta^{r-2\alpha(m/p)} (q')^{(m/p)}.$$

Finally the last terms vanish when summed unless all of n, m, r are multiples of p, in which case we get

$$p^{2k-3-2k+3} q^{n/p} \zeta^{r/p} (q')^{m/p}.$$

Now contributions to $b(n, r, m)$ come from the $a(n', r', m')$ such that the formulae above mention $q^n \zeta^r (q')^m$. So the first term gives $p^{2k-3} a(n/p, r/p, m/p)$, the second $p^{k-2} a(np, r, m/p)$, the third $p^{k-2} \sum_{0 \le \alpha < p} a((n + \alpha r + \alpha^2 m)/p, r + 2\alpha m, mp)$ and the fourth $a(np, rp, mp)$. Done.

$\square$

The next thing I want to do is $T_1(p^2)$ or $T(p^2)$. I don't know which will be most computationally effective. I've done $T_1(p^2)$, here it is.

First a note about degrees. There's a "degree" map from the big ring $\mathbf{Q}[x_0, x_1, x_2, (x_0 x_1 x_2)^{-1}]$ to $\mathbf{Q}$ sending x_0 to 1, x_1 to p and x_2 to p^2. This sends a Hecke operator to the number of single cosets one expects. Evaluating these formulae on our explicit elements corresponding to explicit Hecke operators we see that the degree of $T_0(p^2)$ is $p^6 + p^5 + p^4 + p^3$, the degree of $T_1(p^2)$ is $p^4 + p^3 + p^2 + p$ and the degree of $T(p^2)$ is $p^6 + p^5 + 2p^4 + 2p^3 + p^2 + p + 1$ (the degree of $T_2(p^2)$ is of course 1). Note that equation 3.3.72 of Andrianov gives an explicit coset decomposition for $T(p^2)$ and indeed one can check that the number of cosets one gets is what I said above. Explicitly: for (δ_1, δ_2) with $0 \le \delta_1 \le \delta_2 \le 2$, let d denote the number of cosets that D ranges over, and let b denote the number of cosets that B ranges over. The figures are as follows:

$(0, 0) : d = 1, b = 1$
$(0, 1) : d = p + 1, b = p$
$(0, 2) : d = p^2 + p, b = p^2$
$(1, 1) : d = 1, b = p^3$
$(1, 2) : d = p + 1, b = p^4$
$(2, 2) : d = 1, b = p^6$

And this all adds up. Now this situation breaks up into three subsituations: each one of these double cosets will contribute to $T_i(p^2)$ for precisely one i. The easiest case is $T_2(p^2)$ of course, this is just one coset coming from the $(1, 1)$ case. In general, the way to work out where each coset goes is just to compute the rank of the corresponding space mod p, using the amazing fact that the rank mod p of the matrix used in the definition of $T_i(p^2)$ is $2 - i$.

The way it splits up is as follows: for (δ_0, δ_1) we write $b = b_0 + b_1 + b_2$ with b_i the number of matrices contributing to $T_i(p^2)$ (and hence having rank $2 - i$)

$(0, 0) : d = 1 : b = 1 = 1 + 0 + 0$
$(0, 1) : d = p + 1 : b = p = (p - 1) + 1 + 0$
$(0, 2) : d = p^2 + p : b = p^2 = p^2 + 0 + 0$
$(1, 1) : d = 1 : b = p^3 = (p^3 - p^2) + (p^2 - 1) + 1$

$(1,2): d = p+1 : b = p^4 = (p^4 - p^3) + p^3 + 0$
$(2,2): d = 1 : b = p^6 = p^6 + 0 + 0$

Note in particular that it looks to me that $T_1(p^2)$ will basically only have five "families" of cosets. Let's work them out explicitly:

$(0,1)$: for $D = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & p \end{smallmatrix}\right)$ the matrix is $\begin{pmatrix} p^2 & 0 & 0 & 0 \\ -p\alpha & p & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & p \end{pmatrix}$ and for $D = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ the matrix is $\mathrm{diag}(p, p^2, p, 1)$.

$(1,1)$: $D = \mathrm{diag}(p,p)$ and the matrices are $\begin{pmatrix} p & 0 & a & b \\ 0 & p & b & d \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$ for $0 \le a, b, d < p$ with $ad = b^2 \bmod p$ but not all zero.

$(1,2)$: for $D = \left(\begin{smallmatrix} p & p\alpha \\ 0 & p^2 \end{smallmatrix}\right)$ the matrices are $\begin{pmatrix} p & 0 & 0 & pB \\ -\alpha & 1 & B & \alpha B + D \\ 0 & 0 & p & p\alpha \\ 0 & 0 & 0 & p^2 \end{pmatrix}$ with $0 \le B < p$ and $0 \le D < p^2$. For $D = \mathrm{diag}(p^2, p)$ the matrices are $\begin{pmatrix} 1 & 0 & A & B \\ 0 & p & pB & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$ with $0 \le A < p^2$ and $0 \le B < p$.

Now let's work out what they do to Fourier coefficients. Let's see what happens to $q^N \zeta^R (q')^M$. The constant term ν^{2k-3} is p^{4k-6}.

For $(0,1)$ the α term sends $q^N \zeta^R (q')^M$ to $p^{-k} q^{p^2 N - p\alpha R + \alpha^2 M} \zeta^{pR - 2\alpha M} (q')^M$, and the other term sends it to $p^{-k} q^N \zeta^{pR} (q')^{p^2 M}$.

For $(1,1)$ we get a sum I can't evaluate! If we also include the $T_2(p^2)$ term we get $p^{-2k} c q^N \zeta^R (q')^M$, where c is the sum over a, b, d with $ad = b^2 \bmod p$ of $\zeta_p^{Na+Rb+Md}$. Let's leave it at that.

For $(1,2)$, the α term gives p^{-3k} times something that vanishes unless $p|R$ and $p^2|M$. Assuming this, we get $p^{3-3k} q^{N - \alpha R/p + \alpha^2 M/p^2} \zeta^{R/p - 2\alpha M/p^2} (q')^{M/p^2}$. And the other term is $p^{3-3k} q^{N/p^2} \zeta^{R/p} (q')^M$ if $p^2|N$ and $p|R$, and is zero otherwise.

Finally, we have to decide when this term has degree (n, r, m) as it were. We get

Lemma 2. *If $F = \sum_{n,r,m} a(n,r,m) q^n \zeta^r (q')^m$ then*

$$T_1(p^2)F = \sum_{n,r,m} b(n,r,m) q^n \zeta^r (q')^m,$$

where $b(n,r,m) =$

$$p^{3k-6} \sum_{0 \le \alpha < p} a((n + \alpha r + \alpha^2 m)/p^2, (r + 2\alpha m)/p, m)$$

$$+ p^{3k-6} a(n, r/p, m/p^2)$$

$$+ p^{2k-6} c_p(n,r,m) a(n,r,m)$$

$$+ p^{k-3} \sum_{0 \le \alpha < p} a(n + \alpha r + \alpha^2 m, p(r + 2\alpha m), p^2 m)$$

$$+p^{k-3}a(p^2n,pr,m).$$

Here $c_p(n,r,m)$ is $\sum(\zeta_p)^{na+rb+md}$ where ζ_p is a primitive pth root of unity, and the sum is over all a,b,d with $0 \le a,b,d < p$ and such that a,b,d are not all zero but $b^2 = ad$. And, of course, $a(n,r,m) = 0$ if n,r,m aren't all integers.

Proof. Just done it. $\square$

Now I am going to use Andrianov's (3.3.72) to work out $T(p^2)$ to see if it's any simpler, although my hunch is that it won't be!

For $(0,0)$ we get $\mathrm{diag}(p^2,p^2,1,1)$

For $(0,1)$ we get $\begin{pmatrix} p^2 & 0 & 0 & 0 \\ -\alpha p & p & 0 & d \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & p \end{pmatrix}$ for $0 \le \alpha, d < p$ and then $\begin{pmatrix} p & 0 & a & 0 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for $0 \le a < p$.

The case $(0,2)$ was messy. If $D = \left(\begin{smallmatrix} 1 & \beta \\ 0 & p^2 \end{smallmatrix}\right)$ with $0 \le \beta < p^2$ then we get $\begin{pmatrix} p^2 & 0 & 0 & 0 \\ -\beta & 1 & 0 & d \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & p^2 \end{pmatrix}$ with $0 \le d < p^2$. If $D = \left(\begin{smallmatrix} p & \alpha \\ 0 & p \end{smallmatrix}\right)$ with $1 \le \alpha < p$ then I got $\begin{pmatrix} p & 0 & 0 & B \\ -\alpha & p & B & \delta \\ 0 & 0 & p & \alpha \\ 0 & 0 & 0 & p \end{pmatrix}$ with $0 \le B, \delta < p$. Finally if $D = \left(\begin{smallmatrix} p^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ I got $\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with $0 \le a < p^2$.

Now it's all downhill. For $(1,1)$ I got $\begin{pmatrix} p & 0 & a & b \\ 0 & p & b & d \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$ with $0 \le a,b,d < p$.

For $(1,2)$ and $D = \left(\begin{smallmatrix} p & \alpha p \\ 0 & p^2 \end{smallmatrix}\right)$ I got $\begin{pmatrix} p & 0 & A & \alpha A + pB \\ -\alpha & 1 & B & \alpha B + D \\ 0 & 0 & p & \alpha p \\ 0 & 0 & 0 & p^2 \end{pmatrix}$ with $0 \le A, B < p$ and $0 \le D < p^2$. And for $D = \left(\begin{smallmatrix} p^2 & 0 \\ 0 & p \end{smallmatrix}\right)$ I got $\begin{pmatrix} 1 & 0 & A & B \\ 0 & p & pB & D \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$ with $0 \le A < p^2$ and $0 \le B, D < p$.

Finally, for $(2,2)$ I got $\begin{pmatrix} 1 & 0 & A & B \\ 0 & 1 & B & D \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}$ with $0 \le A,B,D < p^2$.

Next we evaluate how these matrices act on $q^N\zeta^M(q')^R$.

For $(0,0)$ we get $p^{4k-6}q^{p^2N}\zeta^{p^2R}(q')^{p^2M}$.

For $(0,1)$ in the α case we get the following phenomenon which I don't think we've seen before: we get 0 if $p \nmid M$ but if $p|M$ we get a formula that doesn't really involve M/p: we get $p^{3k-5}q^{p^2N-\alpha pR+\alpha^2M}\zeta^{pR-2\alpha M}(q')^M$. In the other case we get the same sort of thing: we get 0 if $p \nmid N$ and $p^{3k-5}q^N\zeta^{pR}(q')^{p^2M}$ otherwise.

For $(0,2)$ in the β case we get $p^{2k-4}q^{p^2N-\beta R+\beta^2M/p^2}\zeta^{R-2\beta M/p^2}(q')^{M/p^2}$ if $p^2|M$ and 0 otherwise. In the α case here are a couple of details: the roots of unity one gets are $\sum_{B,\delta}(\zeta_{p^2})^{pRB-\alpha MB+p\delta M}$ which looks as if it's not even well-defined as B is only going from 0 to p, but it is because the sum vanishes if $p \nmid M$. The sum is zero unless $p|M$ and $p^2|(pR-\alpha M)$, and in this case it's $p^{2k-4}q^{N-\alpha R/p+\alpha^2M/p^2}\zeta^{R-2\alpha M/p}(q')^M$. Finally in the a case it's 0 if $p^2 \nmid N$ and $p^{2k-4}q^{N/p^2}\zeta^R(q')^{p^2M}$ otherwise.

In the $(1,1)$ case it's zero unless p divides all of N,R,M in which case it's $p^{2k-3}q^{N/p}\zeta^{R/p}(q')^{M/p}$.

For $(1,2)$ in the α case it's 0 if $p^2\nmid M$ or $p\nmid N$ or $p\nmid R$, but if all these divisibilities work then it's $p^{k-2}q^{N-\alpha R/p+\alpha^2M/p^2}\zeta^{R/p-2\alpha M/p^2}(q')^M$, and in the other case it's 0 if $p^2\nmid N$ or $p\nmid R$ or $p\nmid M$, and $p^{k-2}q^{N/p^2}\zeta^{R/p}(q')^{M/p}$ otherwise. I will show my working in the α case in case I ever want to reproduce all this! First we have a constant term of p^{4k-6} and a det term of p^{-3k}. Then we have a sum over $0\le A,B<p$ and $0\le D<p^2$ of

$$\left(\left(\begin{smallmatrix} p & 0\\ -\alpha & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} \tau & z\\ z & \tau'\end{smallmatrix}\right)+\left(\begin{smallmatrix} A & \alpha A+pB\\ B & \alpha B+D\end{smallmatrix}\right)\right)\left(\begin{smallmatrix} 1/p & -\alpha/p^2\\ 0 & 1/p^2\end{smallmatrix}\right)$$

which bashes out to

$$\left(\begin{smallmatrix} \tau+A/p & (-\alpha\tau+z+B)/p\\ (-\alpha\tau+z+B)/p & (\alpha^2\tau-2\alpha z+\tau'+D)/p^2\end{smallmatrix}\right).$$

Translating this back into the language of q, ζ and (q') we get

$$q^{N-\alpha R/p+\alpha^2M/p^2}\zeta^{R/p-2\alpha M/p^2}(q')^{M/p^2}$$

times the fudge factor $\sum_{A,B,D}(\zeta_{p^2})^{pAN+pBR+DM}$ and this sum will vanish unless $p|N,p|R,p^2|M$ in which case it's p^4. Now add it all up.

Finally, for $(2,2)$ we get $p^0q^{n/p^2}\zeta^{R/p^2}(q')^{M/p^2}$ if p^2 divides all of N,R,M, and 0 otherwise.

Finally, we add up these contributions and deduce

Lemma 3. *If $F=\sum_{n,r,m}a(n,r,m)q^n\zeta^r(q')^m$ then*

$$T(p^2)F=\sum_{n,r,m}b(n,r,m)q^n\zeta^r(q')^m,$$

where $b(n,r,m)=$

$$p^{4k-6}a(n/p^2,r/p^2,m/p^2)$$
$$+\delta_p(m)p^{3k-5}\sum_{0\le\alpha<p}a((n+\alpha r+\alpha^2m)/p^2,(r+2\alpha m)/p,m)$$
$$+\delta_p(n)p^{3k-5}a(n,r/p,m/p^2)$$
$$+p^{2k-4}\sum_{0\le\beta<p^2}a((n+\beta r+\beta^2m)/p^2,r+2\beta m,mp^2)$$
$$+p^{2k-4}\sum_{1\le\alpha<p}a(n+\alpha r/p+\alpha^2m/p^2,r+2\alpha m/p,m)$$
$$+p^{2k-4}a(p^2n,r,m/p^2)$$
$$+p^{2k-3}a(pn,pr,pm)$$
$$+\delta_{p^2}(m)\delta_p(n+\alpha r+\alpha^2m/p^2)p^{k-2}a(n+\alpha r+\alpha^2m/p^2,pr+2\alpha m/p,m)$$

$$+p^{k-2}a(p^2n, pr, pm)$$
$$+a(p^2n, p^2r, p^2m),$$

where $\delta_a(b)$ is 0 if $a \nmid b$ and 1 if $a|b$.

Remark: in the $1 \le \alpha < p$ case coming from $(0,2)$, note that integrality of $n + \alpha r/p + \alpha^2 m/p^2$, $r + 2\alpha m/p$ and m implies the divisibilities necessary to make the sum of roots of unity non-vanishing.

Proof. Just collect up terms. I'll do the $(1,2)$ α case: we have to solve $n = N - \alpha R/p + \alpha^2 M/p^2$, $r = R/p - 2\alpha M/p^2$ and $m = M$. So we get $M = m$, $R = pr + 2\alpha M/p$ and $N = a + \alpha r + \alpha^2 M/p^2$. This term will contribute if the divisibilities all hold. Now $p^2|M$ translates down to $p^2|m$, which would be implied by integrality of the terms in the a function if $\alpha \neq 0$, but perhaps $\alpha = 0$. This struck me as strange, perhaps I made a mistake. Anyway, assuming $p^2|m$ we see that $p|R$ is now automatic and $p|N$ isn't, so as assume this too.

□

8 Application to computing eigenvalues of l-adic representations.

I will just consider the 4-dimensional representation "conjecturally" associated to a Siegel eigenform. More specifically, take a Siegel form of level N and weight k, and let p be a prime, prime to N, and let l be a prime, and consider the l-adic representation associated to the form, but just as a representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. The status of the write-up of the existence of this representation is not clear to me. But it will be unramified at p and hence determined by the image of Frobenius.

We use notation as in section 5. Let's consider the image of a Frobenius element at p under this spinor representation. The eigenvalues of the matrix should be x_0, x_0x_1, $x_0x_1x_2$ and x_0x_2. The sum of these should be the eigenvalue of $T(p)$. One can do a bit better than the product: one can ask for the norm of the matrix, in the sense of an element of the general symplectic group; this norm will be $x_0^2x_1x_2$, which is p^3 multiplied by the eigenvalue of $T_2(p^2)$. Note that $T_2(p^2)$ is just multiplication by p^{2k-6} so this means that the norm of the matrix is p^{2k-3} (and its determinant is hence p^{4k-6}).

The eigenvalues as a set will be determined by t, ρ_0 and ρ_1 with notation as in section 5; we have seen that in weight k that ρ_0 is p^{2k-3} and t will be the eigenvalue of $T(p)$; the eigenvalues of the matrix will hence be determined once we know ρ_1 which we can work out from $T_1(p^2)$.

Joshua Harris has automated some of these computations for level 1. He computes matrices for $T(p)$ and $T_1(p^2)$. Let's say that he finds a form with eigenvalue λ for $T(p)$ and μ for $T_1(p^2)$, in weight k. What are x_0, x_1 and x_2? We set $\rho_0 = p^{2k-3}$, $\rho_1 = p\mu - (p^2-1)p^{2k-6}$ and $t = \lambda$. Finally, the polynomial

$$X^4 - tX^3 + (\rho_1 + 2\rho_0)X^2 - t\rho_0 X + \rho_0^2$$

comes out as

$$X^4 - \lambda X^3 + (p\mu + (p^3 + p)p^{2k-6})X^2 - \lambda p^{2k-3}X + p^{4k-6}.$$