On p-adic families of automorphic forms

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Abstract

Coleman and Mazur have constructed "eigencurves", geometric objects parametrising certain overconvergent p-adic modular forms. We formulate definitions of overconvergent p-adic automorphic forms for two more classes of reductive groups—firstly for GL_1 over a number field, and secondly for $D^\times$, D a definite quaternion algebra over the rationals. We give several reasons why we believe the objects we construct to be the correct analogue of an overconvergent p-adic modular form in this setting.

1 Introduction

The definition of an overconvergent modular form was formulated in [14] and used in a very powerful way in work of Coleman (see [5], [6], [7] etc) as a key tool for constructing families of modular forms. Using these arguments Coleman was able to resolve some questions of Gouvêa and Mazur about the existence of analytic families of modular forms. In fact, Coleman's ideas gave more: using them, Coleman and Mazur constructed "eigencurves", geometric objects which parametrise finite slope normalised overconvergent modular eigenforms.

There is now evidence that these parameter spaces are just the tip of an iceberg, and that there should exist parameter spaces, or "eigenvarieties", parametrising systems of eigenvalues occurring in p-adic Fréchet spaces of "overconvergent automorphic forms" for a wide class of reductive groups. One way of looking at the formalism is as follows: firstly there should be some appropriate set of classical infinity types (for example, the regular algebraic ones), which naturally form a subset of a rigid analytic space of "p-adic infinity types", or "weights" for short. We call such a space a *weight space*, we call points of this space *weights*, and we call the subset mentioned above the set of *classical weights*. If κ is a weight, then the space of overconvergent automorphic forms of some fixed level U and weight κ should be a (typically infinite-dimensional) p-adic vector space that, if κ is classical, will naturally contain the finite-dimensional space of classical automorphic forms of this weight (after one has chosen isomorphisms $\mathbf{C} \cong \mathbf{C}_p$ and so on). Furthermore, the spaces should naturally p-adically interpolate the classical spaces of forms as the infinity type varies in weight space. Note that in this context, as was observed in the 1970s by Serre and Katz, the notion of a weight should include both an infinity type and also a level structure at the prime p.

Now fix a level structure U prime to p. Then there should be a second rigid space, the eigenvariety associated to this data—this will be a rigid-analytic space equipped with a map to weight space, such that the fibre above a weight will be equal to the spectrum of a Hecke algebra acting on the space of "finite slope" overconvergent forms of that weight, and "tame level" U. In this perspective, a Hecke operator becomes a map from the eigenvariety to the affine line, sending a point to the eigenvalue of the corresponding eigenform.

There should be a family of p-adic Galois representations living on the eigenvariety, and the eigenvariety should in general be strongly connected to the deformation theory of such Galois representations. Moreover, the eigenvariety should perhaps be the natural domain for the special values of p-adic L-functions, in analogy with the role of the collection of Riemann surfaces that Tate uses to great effect in the study of classical L-functions in his thesis (see p314 of [3]). These sides of the story will not be treated in this paper, however.

One interesting application of the existence of these eigenvarieties is that one can re-formulate cases of Langlands functoriality in this setting as simply predicting the existence of a morphism between two such parameter spaces that intertwines Hecke operators appropriately.

It is not currently clear to the author in what generality one would expect these parameter spaces to exist, but this may be due to a large extent to the fact that the author is certainly no expert in the theory of automorphic forms. On the other hand, there do seem to be some cases where this philosophy does not apply. For example, we know of no evidence to suggest that Maass forms can be p-adically interpolated in such a way, although perhaps Mazur's observation that it seems harder to deform even Galois representations than odd ones might be evidence to suggest that the associated rigid spaces may all be 0-dimensional. On the other hand, there is a growing body of evidence to suggest that in many other common cases, non-trivial parameter spaces do exist. The author is optimistic that, for example, if the reductive group in question is compact mod centre at infinity then almost all the techniques now exist for constructing such parameter spaces. See the thesis of G. Chenevier [4], forthcoming work [2] of the author and also future joint work of the author and Chenevier, where the constructions will be given in many cases. Unfortunately, the works cited above are perhaps daunting to read, relying on many technical computations in representation theory, as well as a generalisation of much of Coleman's theory of families of p-adic Banach spaces to families of dimension greater than one.

The purpose of this paper however is to give an explicit construction of the parameter spaces in two of the simplest cases: firstly in the case of GL_1 over a number field, and secondly in the case of $D^\times$, D a definite quaternion algebra over the rationals. The motivation for explaining these cases explicitly is several-fold. Firstly, the theory for GL_1 drops out from class field theory and known results about p-adic Galois representations, and is excellent motivation for believing that these parameter spaces exist in some generality. One avoids having to generalise Coleman's theorems because the Banach spaces in question are finite-dimensional and everything is easy. Note that already the dimension of the parameter spaces in this case depends on the truth of Leopoldt's conjecture, but not knowing the dimension of the spaces constructed here does not seem to lead to any problems in the theory. Secondly, the representation theory involved in the GL_1 case is trivial, so one sees a concrete example of the theory without having to get too bogged down in technicalities.

One motivation for the construction in the definite quaternion algebra case is that it is easy to make all the representation theory very explicit, thus avoiding the technical computations in [4] where analogous results are proved for compact forms of GL_n. Another reason is that by the Jacquet-Langlands theorem, classical automorphic forms in this setting will correspond to classical modular forms, and overconvergent automorphic forms in this setting should correspond to overconvergent p-adic modular forms (in fact this has recently been proved by Chenevier: see [4]). The families one constructs are one-dimensional, so one does not need to generalise Coleman's work either. We refer to Theorem 1 below for some of the results proved by Coleman and others about classical and overconvergent modular forms. On the other hand, we show in this paper that the analogues of these theorems in this setting are, for the most part, elementary consequences of the definitions. This gives weight to our hope that our definitions are the right ones in this setting.

Some of this work in this paper was done during the semestre automorphe at the IHP in Paris in 2000, and some was done during a visit to Paris 13 funded by the Arithmetic Algebraic Geometry RTN network. The author thanks all these institutions. The results for GL_1 in this paper were presented at the conference on modular curves at the CRM in Barcelona, and the author thanks the organisers of this conference for giving him the opportunity to speak there. The author also thanks the referee for several useful comments and observations.

Note added in proof: recent preprints of Emerton seem to give the construction of eigenvarieties in a very wide class of situations, using more representation-theoretic techniques.

2 The case of GL_1 over a number field

Throughout this paper, $p > 0$ will be a prime, and K will denote a field which is complete with respect to a non-trivial non-Archimedean valuation, and whose residue field has characteristic p. The rigid spaces we construct will be rigid spaces over K. We will consider K as being fixed—the interested reader can verify that all rigid spaces in this paper are quasi-separated and their definitions commute with base change if K' is a complete extension of K. Up to and including Lemma 2, K may have characteristic 0 or p, but after Lemma 2 and for the rest of the paper we assume that the characteristic of K is zero. From section 3 onwards, we assume furthermore that K is a complete subfield of $\mathbf{C}_p$, but this is only for notational convenience; one has to do little more than to replace p by $|p|^{-1}$ on many occasions to deal with the general case.

Let $L/\mathbf{Q}$ be a number field of degree d, and let $\mathcal{O}_L$ denote the integers of L.

Let L_∞ denote $L \otimes_{\mathbf{Q}} \mathbf{R}$ and $(L_\infty^\times)^\circ$ the identity component of the topological group $L_\infty^\times$. Similarly, let L_p denote $L \otimes_{\mathbf{Q}} \mathbf{Q}_p$, let $\mathcal{O}_{L,p}$ denote $\mathcal{O}_L \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and let $\mathcal{O}_{L,p}^\times$ denote the topological group of units in $\mathcal{O}_{L,p}$.

We would like to formulate a definition of an overconvergent p-adic automorphic form for the group $\mathrm{Res}_{L/\mathbf{Q}} \mathrm{GL}_1$, defined over K. Associated to an overconvergent eigenform we should expect a one-dimensional p-adic Galois representation. On the other hand, any Hodge-Tate one-dimensional p-adic representation of $\mathrm{Gal}(\overline{L}/L)$ comes from an algebraic Grössencharacter, and hence one should suspect that for classical weights, the overconvergent eigenforms should be in bijection with the classical eigenforms. In fact the main problem for GL_1 is simply that of constructing weight space and we present the natural definition below.

Lemma 1. *Let R be an affinoid algebra over K. For there to exist a continuous group homomorphism $\mathbf{Z}_p \to R^\times$ sending 1 to $1 + r \in R$, it is necessary and sufficient that r should be topologically nilpotent, and when this is the case, the group homomorphism sending 1 to $1 + r$ is unique.*

Proof. Fix a complete K-algebra norm on R inducing the topology on R. If there is a continuous group homomorphism $\mathbf{Z}_p \to R^\times$ sending 1 to $1 + r$ then $(1 + r)^{p^n} \to 1$ as $n \to \infty$ in R, and the same is true in $R/\mathfrak{m}$ for $\mathfrak{m}$ any maximal ideal of R. This means that modulo $\mathfrak{m}$, the residue norm of r must be less than 1 (look at the residue field). Hence r is topologically nilpotent by Proposition 6.2.3.2 of [1]. Conversely, if r is topologically nilpotent, then $|r^n| \to 0$ as $n \to \infty$ and hence an easy argument using the binomial theorem shows that $(1+r)^{p^N}$ tends to 1 as $N \to \infty$, which suffices to construct the group homomorphism $\mathbf{Z}_p \to R^\times$, and uniqueness follows because of continuity. □

Now let X be an arbitrary rigid space, and let $\mathcal{O}(X)$ denote the global sections of the structure sheaf on X. If H is a topological group, then we say that a group homomorphism $H \to \mathcal{O}(X)^\times$ is *continuous* if for all admissible affinoid subdomains U of X, the resulting map $H \to \mathcal{O}(U)^\times$ is continuous. The lemma above shows that there is a canonical functorial bijection between the continuous homomorphisms $\mathbf{Z}_p \to \mathcal{O}(X)^\times$ and the maps of rigid spaces $X \to \Delta$, where Δ denotes the open disc of radius 1 and centre 1, considered as a subspace of rigid affine 1-space.

Let $\mathbf{G}_m$ denote the rigid space over K associated to the affine scheme $\mathbf{A}^1\backslash\{0\}$. We now explain an elementary but key construction. Let H be an abelian profinite group which contains an open subgroup isomorphic to $(\mathbf{Z}_p)^n$ for some $n \geq 0$. Let $\mathrm{Hom}(H, \mathbf{G}_m)$ denote the functor on rigid spaces over K sending a rigid space X to the set of continuous group homomorphisms $H \to \mathcal{O}(X)^\times$.

Lemma 2. *(i) The functor* $\mathrm{Hom}(H, \mathbf{G}_m)$ *is represented by a quasi-separated rigid space over K, also denoted* $\mathrm{Hom}(H, \mathbf{G}_m)$.

(ii) The space $\mathrm{Hom}(H, \mathbf{G}_m)$ *is a union of finitely many open n-balls and naturally has the structure of a group object in the category of rigid spaces over K.*

(iii) If $H \to J$ is a surjective morphism of profinite groups with finite kernel, then the canonical map $\mathrm{Hom}(J, \mathbf{G}_m) \to \mathrm{Hom}(H, \mathbf{G}_m)$ *is a closed immersion. If furthermore the size of the kernel of $H \to J$ is non-zero in K (for example, if the characteristic of K is 0), then the canonical map* $\mathrm{Hom}(J, \mathbf{G}_m) \to \mathrm{Hom}(H, \mathbf{G}_m)$ *is a closed and open immersion, identifying* $\mathrm{Hom}(J, \mathbf{G}_m)$ *with a union of components of* $\mathrm{Hom}(H, \mathbf{G}_m)$.

(iv) If $H \to J$ is an injective morphism of profinite groups with finite cokernel, then the canonical map $\mathrm{Hom}(J, \mathbf{G}_m) \to \mathrm{Hom}(H, \mathbf{G}_m)$ *is finite and flat, of degree equal to $[J : H]$. If furthermore $[J : H] \neq 0$ in K then the map is etale.*

Proof. Note firstly that if H_1 and H_2 satisfy the hypotheses put on H above, and $\mathrm{Hom}(H_1, \mathbf{G}_m)$ and $\mathrm{Hom}(H_2, \mathbf{G}_m)$ are representable, then one can represent $\mathrm{Hom}(H_1 \times H_2, \mathbf{G}_m)$ by the product $\mathrm{Hom}(H_1, \mathbf{G}_m) \times \mathrm{Hom}(H_2, \mathbf{G}_m)$. Furthermore, if H is as in the lemma, then by the structure theorem for topologically finitely-generated abelian profinite groups, H is isomorphic to the product of a finite group and a group isomorphic to $(\mathbf{Z}_p)^n$.

(i) By the remarks above, we are reduced to representing the functor in the following two cases: the case of finite cyclic H, and the case $H \cong \mathbf{Z}_p$. If H is finite cyclic of order n then the rigid space associated to the affinoid algebra $K\langle T \rangle / (T^n - 1)$ is easily shown to do the trick. If on the other hand $H \cong \mathbf{Z}_p$, then by Lemma 1 and the remarks following it, the open disc with centre 1 and radius 1 over K represents the functor in question.

(ii) By the construction of the representing space above, we see it is a product of rigid spaces each of which is either finite or an open unit ball, and the results follow easily.

(iii) Here one only need observe that H is isomorphic to the product of a free $\mathbf{Z}_p$-module and a finite group, and the kernel of the map $H \to J$ is contained in this finite group. One now easily reduces to the case where H is finite, where the result is easy.

(iv) One reduces easily to the case where J/H is finite cyclic of degree n for some $n \geq 1$. In this case the cover $\mathrm{Hom}(J, \mathbf{G}_m)$ of $\mathrm{Hom}(H, \mathbf{G}_m)$ is defined by taking the nth root of some function f on $\mathrm{Hom}(H, \mathbf{G}_m)$, and the fact that $|f - 1| < 1$ for any residue norm means that f is a unit. Hence the covering is finite and flat, and moreover it is etale if n is prime to the characteristic of K. □

We now use this lemma to construct a "weight space". Assume for the rest of this paper that the characteristic of K is zero. If Γ is a subgroup of $\mathcal{O}_L^\times$ of finite index, and $\overline{\Gamma}$ denotes the closure of Γ in $\mathcal{O}_{L,p}^\times$, then the quotient $H_\Gamma := \mathcal{O}_{L,p}^\times / \overline{\Gamma}$ satisfies the hypotheses of the lemma. Hence one has a rigid space $\mathcal{W}_\Gamma := \mathrm{Hom}(H_\Gamma, \mathbf{G}_m)$ defined over K. If $\Delta \subseteq \Gamma$ is a subgroup of finite index, then the corresponding surjection $H_\Delta \to H_\Gamma$ has kernel of finite order $[\overline{\Gamma} : \overline{\Delta}]$ and hence by part (iii) of Lemma 2 there is a closed and open immersion $\mathcal{W}_\Gamma \to \mathcal{W}_\Delta$, identifying the former with a union of components of the latter.

Definition. *We define* weight-space $\mathcal{W}$ *to be the direct limit* $\lim_{\to} \mathcal{W}_\Gamma$, *as* Γ

varies over the set of subgroups of finite index in $\mathcal{O}_L^\times$, partially ordered by inclusion.

The space $\mathcal{W}$ is a rigid space defined over K. It has a group structure, which we in fact shall never use. By definition, a K-point in $\mathcal{W}$ corresponds to a continuous group homomorphism $\mathcal{O}_{L,p}^\times \to K^\times$ which vanishes on some subgroup of finite index in $\mathcal{O}_L^\times$.

Let $\sigma_1, \ldots, \sigma_d$ denote the d embeddings $L \to \mathbf{C}_p$. Each such embedding gives a continuous group homomorphism $\mathcal{O}_{L,p}^\times \to \mathbf{C}_p^\times$, which we also denote by σ_i.

Definition. *We say that a weight in $\mathcal{W}(K)$ is* classical *if there are a collection $(n_1, n_2, \ldots, n_d)$ of integers, and a subgroup H of finite index of $\mathcal{O}_{L,p}^\times$, such that the corresponding group homomorphism κ, when restricted to H, equals*

$$(\sigma_1)^{n_1}(\sigma_2)^{n_2}\ldots(\sigma_d)^{n_d}.$$

Choose a K-point of $\mathcal{W}_K$ and let κ denote the corresponding map $\mathcal{O}_{L,p}^\times \to K^\times$. If S is a finite set of places of $\mathbf{Q}$, let $\mathbf{A}_L^S$ denote the adeles of L away from the places in L above the places in S.

Set $S = \{p, \infty\}$ once and for all, and let U be a compact open subgroup of $(\mathbf{A}_L^S)^\times$. We refer to such a subgroup U as a *tame level.*

Let G denote the group $\mathrm{Res}_{L/\mathbf{Q}} \mathrm{GL}_1$.

Definition. *An* overconvergent automorphic eigenform *for G, of weight κ and tame level U, defined over K, is a continuous group homomorphism $L^\times\backslash\mathbf{A}_L^\times \to K^\times$ which contains $U.(L_\infty^\times)^\circ$ in its kernel, and whose restriction to $\mathcal{O}_{L,p}^\times$ equals κ. An* overconvergent automorphic form *for G, of weight κ and tame level U, defined over K, is a continuous function $f : L^\times\backslash\mathbf{A}_L^\times \to K$ such that $f(gu) = f(g)$ for all $u \in U.(L_\infty^\times)^\circ$, the K-vector space generated by all the $\mathbf{A}_L^\times$-translates of f is finite-dimensional, and such that, after a finite base extension of K if necessary, there is a basis of this finite-dimensional space consisting of overconvergent automorphic eigenforms for G of weight κ and tame level U.*

Note that any continuous group homomorphism $L^\times\backslash\mathbf{A}_L^\times \to K$ will automatically contain $(L_\infty^\times)^\circ$ and some compact open subgroup of $(\mathbf{A}_L^S)^\times$ in its kernel, and hence its restriction to $\mathcal{O}_{L,p}^\times$ will automatically be a weight, in the sense that it must vanish on a subgroup of $\mathcal{O}_L^\times$ of finite index.

Note that there are only finitely many overconvergent automorphic eigenforms of a given weight and tame level, because $L^\times\backslash\mathbf{A}_L^\times/\mathcal{O}_{L,p}^\times U(L_\infty^\times)^\circ$ is finite. By the main theorem of global class field theory, to an overconvergent eigenform defined over K there is associated a continuous one-dimensional representation $\mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_1(K)$. If K is a finite extension of $\mathbf{Q}_p$ and $\kappa \in \mathcal{W}(K)$, then this associated Galois representation will be Hodge-Tate if and only if κ is classical. In fact, for classical κ, the space we have constructed perhaps deserves to be thought of as either the space of p-adic, overconvergent p-adic, or classical forms defined over K.

Note that Weil associated a Hodge-Tate p-adic Galois representation to any algebraic Grössencharacter $L^\times\backslash\mathbf{A}_L^\times \to \mathbf{C}^\times$ and this p-adic Galois representation gives rise to an overconvergent automorphic eigenform (although of course one does not need class field theory here—Weil's argument directly constructs an overconvergent automorphic eigenform from a classical one). The weight of such an eigenform is classical, and the corresponding integers $(n_1, n_2, \ldots, n_d)$ are the Hodge-Tate weights of the associated Galois representation, if one normalises the isomorphism of global class field theory appropriately.

It is easy to see, given what we have done, that these definitions interpolate naturally. More precisely, if V is an affinoid subdomain of $\mathcal{W}$ with $\mathcal{O}(V) = R$, then the inclusion $V \to \mathcal{W}_K$ gives us, from the functorial property of $\mathcal{W}_K$, a continuous group homomorphism $\mathcal{O}_{L,p}^\times \to R^\times$. Now for a tame level U one can define an overconvergent automorphic eigenform of weight V as simply being a continuous group homomorphism $f : L^\times\backslash\mathbf{A}_L^\times \to R^\times$ containing $U.(L_\infty^\times)^\circ$ in its kernel, such that the restriction of f to $\mathcal{O}_{L,p}^\times$ is the group homomorphism $\mathcal{O}_{L,p}^\times \to R^\times$ corresponding to the map $V \to \mathcal{W}_K$, and one can give a similar definition of an overconvergent automorphic form of weight V. These spaces should of course be thought of as interpolating the spaces of overconvergent forms of weight κ introduced above, as κ varies through V.

Let U be a fixed tame level, and let Γ denote the intersection

$$L^\times \cap U.\mathcal{O}_{L,p}^\times(L_\infty^\times)^\circ.$$

Then Γ is a subgroup of $\mathcal{O}_L^\times$ of finite index. We will now define an "eigenvariety" of tame level U, which will be a finite cover of the weight space $\mathcal{W}_\Gamma$ introduced earlier. Its definition is simple, given what we have already. Let H denote the quotient of the group $L^\times\backslash\mathbf{A}_L^\times$ by the closure of the image of $U.(L_\infty^\times)^\circ$. Then the map $\mathcal{O}_{L,p}^\times \to H$ is continuous with finite cokernel, and hence H satisfies the hypotheses of Lemma 2.

Definition. *We define the eigenvariety of tame level U to be the rigid space*

$$\mathrm{Hom}(H, \mathbf{G}_m).$$

Let $\overline{\Gamma}$ denote the closure of Γ in $\mathcal{O}_{L,p}^\times$. Then the natural map $\mathcal{O}_{L,p}^\times/\overline{\Gamma} \to H$ is injective and the cokernel is finite. The corresponding map of rigid spaces $\mathrm{Hom}(H, \mathbf{G}_m) \to \mathcal{W}_\Gamma$ is finite flat of degree equal to the order of this cokernel, by part (iv) of Lemma 2. Note that it also follows from this lemma that the eigenvariety is a finite union of open balls, and that the fibre of a point κ in weight space is canonically the set of overconvergent automorphic forms of weight κ and tame level U. The eigenvariety itself can be thought of as a parameter space interpolating overconvergent eigenforms of varying weights. Finally, the Hecke operator corresponding to a uniformiser at a place $\mathfrak{q}$ of L manifests itself as the evaluation homomorphism from the eigenvariety to $\mathbf{G}_m$ corresponding to evaluation at this uniformiser.

In this setting, the eigenvariety is smooth and has finitely many components. It still appears to be an open question as to whether the Coleman-Mazur eigencurve has finitely or infinitely many components (indeed, almost every natural question about the Coleman-Mazur eigencurve is still open).

3 Classical and overconvergent modular forms: Results of Coleman and others.

In this section we briefly summarise (see Theorem 1 below) some of the main results from the theory of classical and overconvergent modular forms. Our goal in the rest of this paper is to propose analogous definitions in the setting of automorphic forms for $D^\times$, D a definite quaternion algebra over $\mathbf{Q}$, and then to prove analogues of the results in Theorem 1, although we shall only sketch the construction of the eigencurve in this setting, and the construction will be made under a minor but unnecessary technical restriction, which we shall remove in forthcoming work.

Let p be prime, and let N be a positive integer prime to p. Let K be a complete subfield of $\mathbf{C}_p$ (this is just for notational convenience; all of this works for a general K of characteristic zero, complete with respect to a non-trivial non-archimedian valuation and with residue field of characteristic p), and for $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ let $S_k(\Gamma)$ denote the space of classical cusp forms of level Γ and weight k defined over K. If $\Gamma = \Gamma_1(N)$ then we refer to this space as the space of classical cusp forms of level N and weight k defined over K. For $r = p^{-t}$ with $t \in \mathbf{Q}$ and $0 \leq t < p/(p+1)$, let $\mathbf{S}_k(\Gamma_1(N); r)$ denote the p-adic Banach space of r-overconvergent cusp forms of tame level N and weight k, that is, at least for $N \geq 5$, the cuspidal sections of ω^k on the rigid subspace of $X_1(N)$ obtained by removing open discs of radius r above every supersingular point in characteristic p (a more precise definition may be found in, for example, [7] and [5]. We omit the more precise definitions as we shall never be using them). Let $\mathbf{S}_k(\Gamma_1(N))$ denote the space of forms which overconverge as far as one can reasonably ask, that is, sections of ω^k on the wide-open subspace of $X_1(N)$ obtained by removing closed discs of radius $p^{-p/(p+1)}$ in each supersingular disc. By definition, $S_k(\Gamma_1(N)) \subset \mathbf{S}_k(\Gamma_1(N)) \subset \mathbf{S}_k(\Gamma_1(N); r)$ and if $r < s$ then $\mathbf{S}_k(\Gamma_1(N); r) \subset \mathbf{S}_k(\Gamma_1(N); s)$. We remind the reader of some theorems about these objects (again we shall be slightly sketchy, and refer the reader to the original papers for more precise statements)

Theorem 1. *1. If $f \in S_k(\Gamma_1(N) \cap \Gamma_0(p^n))$ then $f \in \mathbf{S}_k(\Gamma_1(N); r)$ for some appropriate r.*

2. If $f \in \mathbf{S}_k(\Gamma_1(N); r)$ and $U_p f = \lambda f$ for some non-zero λ, then $f \in \mathbf{S}_k(\Gamma_1(N))$.

3. For $k \geq 1$, there is a map $\theta^{k-1} : \mathbf{S}_{2-k}(\Gamma_1(N)) \to \mathbf{S}_k(\Gamma_1(N))$ which on q-expansions is $\left(q\frac{d}{dq}\right)^{k-1}$, and whose cokernel is finite-dimensional.

4. Say $0 \neq f \in \mathbf{S}_k(\Gamma_1(N); r)$ and $U_p f = \lambda f$ for some non-zero λ. If $v_p(\lambda) <$

$k-1$, then $f \in S_k(\Gamma_1(N))$. On the other hand, if $v_p(\lambda) > k-1$ then $f \notin S_k(\Gamma_1(N))$.

5. *Any finite slope overconvergent eigenform is part of a family of constant slope, in some precise sense.*

6. *Finite slope normalised overconvergent eigenforms of tame level N are parametrised by a rigid-analytic geometric object, the eigencurve of tame level N (at least if $p > 2$ and $N = 1$).*

Proof. Parts 1 and 2 are classical. Part 1 is proved in [14] for $n = 1$ and in [10] for general n. Part 2 is also proved in [14]. Parts 3 and 4 are related and deeper, and are both proved in [5]. Part 5 is proved in [6] and part 6 in [7]. Note also that the constructions in [7] should also go through for general p and N, although nothing is published in this generality as yet (see [2] and also forthcoming work of Emerton, however).

□

As already explained, we will prove analogues of all these results in the setting of automorphic forms on definite quaternion algebras over the rationals. The main question to be solved here is that of finding the correct definition of an overconvergent automorphic form in this setting. The problem with the classical definition is that it is inherently geometric, and relies (amongst other things) on the fact that modular curves are one-dimensional. This causes problems when trying to construct analogues of the Coleman-Mazur construction for other reductive groups, although some progress has been made by Goren and others in the case of Hilbert modular forms. See also forthcoming work of Nevens, and recent preprints of Abbes-Mokrane and Kisin-Lai, which should hopefully culminate in a geometric construction of p-adic parameter spaces for Hilbert modular forms.

In the setting of groups which are compact mod centre at infinity, the problem is rather different: now one cannot remove canonical small regions from the moduli space because it is 0-dimensional, and it appears that a completely different approach is necessary. Our approach is very much inspired by a preprint [9] of Stevens and involves using an "Eichler-Shimura" philosophy which re-interprets the theory of modular forms in a much more combinatorial way, which seems to be more amenable to generalisation. One consequence of this is that the proofs of the analogues of the results above in this setting are much easier, requiring no hard algebraic geometry.

One natural question that can be raised is what relation there is between the eigencurves constructed here and those of Coleman-Mazur. Is there a map between them "explaining" the Jacquet-Langlands correspondence? In fact Chenevier in [4] has recently constructed such a map, although his construction assumes the classical Jacquet-Langlands result and hence does not give another proof of it.

4 Definitions and classical forms

Let p be a fixed prime. For $\alpha \geq 1$ an integer, let $\mathbf{M}_\alpha$ be the monoid consisting of 2 by 2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with a, b, c, $d \in \mathbf{Z}_p$ and such that $p^\alpha \mid c$, $p \nmid d$ and $ad - bc \neq 0$. Let D be a definite quaternion algebra over $\mathbf{Q}$ with discriminant δ prime to p. Fix once and for all a maximal order $\mathcal{O}_D$ of D and an isomorphism $\mathcal{O}_D \otimes B \cong \mathrm{M}_2(B)$, where $B = \lim_{\leftarrow}(\mathbf{Z}/M\mathbf{Z})$, the limit being taken over all integers M prime to δ. This isomorphism induces isomorphisms $\mathcal{O}_D \otimes \mathbf{Z}_l \cong \mathrm{M}_2(\mathbf{Z}_l)$ and $\mathcal{O}_D \otimes \mathbf{Q}_l \cong \mathrm{M}_2(\mathbf{Q}_l)$ for all primes $l \nmid \delta$, and we will henceforth identify these rings with one another. Let $\mathbf{A}_f$ denote the finite adeles over $\mathbf{Q}$ and define $D_f = D \otimes_{\mathbf{Q}} \mathbf{A}_f$. Then D_f can be thought of as the restricted product over all primes l of $D \otimes \mathbf{Q}_l$ and in particular if $g \in D_f$ then the component g_p of g at p can be regarded as an element of $\mathrm{M}_2(\mathbf{Q}_p)$.

Definition. *If U is an open compact subgroup of $D_f^\times$ and $\alpha \geq 1$ then we say that U has* wild level $\geq p^\alpha$ *if the projection of U onto $\mathrm{GL}_2(\mathbf{Q}_p)$ is contained in $\mathbf{M}_\alpha$.*

We offer the following as examples. If M is any integer prime to δ, then define $U_0(M)$ (resp. $U_1(M)$) to be the compact open in $D_f^\times$ formed as a product $\prod_l U_l$ where $U_l = (\mathcal{O}_D \otimes \mathbf{Z}_l)^\times$ for $l \mid \delta$, and U_l is the matrices which are of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$) mod $l^{v_l(M)}$ for all other l. If p^α divides M then the projection of $U_1(M)$ to the factor at p is contained in $\mathbf{M}_\alpha$ and hence U has wild level $\geq p^\alpha$. At the other extreme, we say that a compact open U is "prime to p" if U can be written as $U' \times \mathrm{GL}_2(\mathbf{Z}_p)$ where U' is a compact open of $(D \otimes \mathbf{A}_{f,p})^\times$, where $\mathbf{A}_{f,p}$ denotes the adeles with trivial components at infinity and p. As examples, if N is a positive integer prime to $p\delta$ then $U_0(N)$ and $U_1(N)$ have level prime to p. We refer to the subgroups U' above as "tame levels".

Let $\alpha \geq 1$ be an integer, and let U be a compact open of wild level $\geq p^\alpha$. If R is any commutative ring and A is an R-module with an R-linear right action of $\mathbf{M}_\alpha$, then define the R-module $\mathcal{L}(U, A)$ by

$$\mathcal{L}(U, A) = \left\{ f : D_f^\times \to A \, : \, f(dgu) = f(g)u_p \right\},$$

where $d \in D^\times$ (embedded diagonally in $D_f^\times$), $g \in D_f^\times$ and $u \in U$. This object looks rather terrifying if one is not familiar with this kind of thing, but in fact it is well-known that if one writes $D_f^\times$ as a disjoint union of double cosets $D_f^\times = \coprod_i D^\times d_i U$ then this union is finite, of size n say, and hence any $f \in \mathcal{L}(U, A)$ is determined uniquely by the finite amount of data $f(d_1)$, $f(d_2)$, $\ldots f(d_n)$. More precisely it can be shown that if $\Gamma_i = d_i^{-1} D^\times d_i \cap U$ then Γ_i is a finite group, and then an easy formal argument shows that the map

$$\mathcal{L}(U, A) \to \bigoplus_{i=1}^{n} A^{\Gamma_i} \tag{1}$$

sending f to $\big(f(d_i)\big)_{1 \leq i \leq n}$ is an isomorphism of R-modules. In particular, if U is chosen such that all the groups Γ_i are trivial, then the map $\mathcal{L}(U, A) \to A^n$

sending f to $\big(f(d_1), f(d_2), \ldots\big)$ is an isomorphism of R-modules. Triviality of all the Γ_i is a common occurrence—if $\gamma \in \Gamma_i$ is non-trivial then the characteristic polynomial of γ must be of the form $X^2 - aX + 1$ where $a \in \{-2, -1, 0, 1\}$, with $a = -2$ occurring iff $\gamma = -1$. Hence if $U \subseteq U_1(M)$ for some $M \geq 4$, for example, then all the Γ_i are trivial. The non-triviality of the groups Γ_i can be dealt with if one extends some of Coleman's theory on families of compact operators—see forthcoming work [2] of the author. For simplicity we shall here assume that all Γ_i are trivial whenever this eases the exposition. The more sophisticated reader might be happier thinking of $\mathcal{L}(U, A)$ in the general case as being the global sections of the sheaf corresponding to A on the stack $D^\times \backslash D_f / U$.

We recall the classical definition of the space of automorphic forms of level U and weight k. Strictly speaking these are not classical forms because we have shifted the "weight" action from ∞ to p, so one should perhaps think of the spaces below as being analogues of modular forms with coefficients in K. It is an elementary exercise (see for example the argument around equation (2) on p443 of [8]) to verify, however, that if there is an embedding of fields $K \to \mathbf{C}$ then the K-spaces we are about to define, when tensored up to $\mathbf{C}$, become isomorphic as Hecke modules to spaces of classical forms.

Recall that K is a complete subfield of $\mathbf{C}_p$. Let L_k denote the space of polynomials in one variable z, of degree at most $k-2$, and with coefficients in K. Define an action of $\mathbf{M}_1$ (and hence of $\mathbf{M}_\alpha$ for all $\alpha \geq 1$) by, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_1$ and $h \in L_k$,

$$(h|\gamma)(z) = (cz + d)^{k-2} h\left(\frac{az + b}{cz + d}\right).$$

If U is a compact open subgroup of $D_f^\times$ of wild level $\geq p$, we shall refer to the K-vector space $S_k^D(U) := \mathcal{L}(U, L_k)$ as the space of classical automorphic forms of weight k and level U for D, defined over K. If $U = U_1(M)$ for some integer M which is a multiple of p and prime to δ as usual, then we refer to $S_k^D(U_1(M))$ as the space of classical automorphic forms for D of level M and weight k, defined over K. If K contains all the $\phi(M)$th roots of unity then this space decomposes as a direct sum

$$S_k^D(U_1(M)) = \bigoplus_{\varepsilon} S_k^D(U_1(M))(\varepsilon)$$

where the sum is over all characters $\varepsilon : (\mathbf{Z}/M\mathbf{Z})^\times \to K^\times$, and $S_k^D(U_1(M))(\varepsilon)$ is defined to be

$$\left\{ f : D_f^\times \to L_k \;:\; f(dgu) = \varepsilon(u) f(g) u_p \text{ for all } d \in D^\times, u \in U_0(M) \right\}.$$

Here ε is considered as a character of $U_0(M)$ via the map $U_0(M) \to (\mathbf{Z}/M\mathbf{Z})^\times$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to d mod M.

It is more convenient to split up a character into its p-power part and its p-primary part. If $M = Np^n$ with $n \geq 1$ and N prime to p, and we continue to assume that K contains all $\phi(M)$th roots of unity, then one can write $\varepsilon : (\mathbf{Z}/M\mathbf{Z})^\times \to K^\times$ as a product $\varepsilon^p \varepsilon_p$, where ε^p has level N and ε_p has level p^n. Let $\alpha \geq 1$ be minimal such that ε_p factors through $(\mathbf{Z}/p^\alpha\mathbf{Z})^\times$. One can then define a "weight-character (k,ε_p)"-action of $\mathbf{M}_\alpha$ on L_k by, for $h \in L_k$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_\alpha$,

$$(h|\gamma)(z) = (cz+d)^{k-2}\varepsilon_p(d) h\left(\frac{az+b}{cz+d}\right).$$

Let L_{k,ε_p} denote L_k equipped with this action. If U is any compact open of level prime to p, then it makes sense to define $S_k^D(U \cap U_1(p^n))(\varepsilon_p) := \mathcal{L}(U \cap U_0(p^n), L_{k,\varepsilon_p})$ and we see that

$$\bigoplus_{\varepsilon^p : (\mathbf{Z}/N\mathbf{Z})^\times \to K^\times} S_k^D(U_1(M))(\varepsilon^p \varepsilon_p) = S_k^D(U_1(M))(\varepsilon_p),$$

$$\bigoplus_{\varepsilon_p : (\mathbf{Z}/p^n\mathbf{Z})^\times \to K^\times} S_k^D(U_1(M))(\varepsilon_p) = S_k^D(U_1(M)),$$

and more generally

$$\bigoplus_{\varepsilon_p : (\mathbf{Z}/p^n\mathbf{Z})^\times \to K^\times} S_k^D(U \cap U_1(p^n))(\varepsilon_p) = S_k^D(U \cap U_1(p^n))$$

if U has level prime to p. This philosophy, of treating characters at p differently to characters of level prime to p, by amalgamating the character at p with the weight, is already present in the "classical" theory of p-adic modular forms, and we also saw it in the GL_1 case.

Now let us return to the setting of a compact open U of wild level $\geq p^\alpha$, and a general right $\mathbf{M}_\alpha$-module A. The space $\mathcal{L}(U,A)$ comes equipped with certain standard operators, which we now define. Firstly say $\eta \in D_f^\times$ such that $\eta_p \in \mathbf{M}_\alpha$. If $f : D_f^\times \to A$ then define $f|\eta : D_f^\times \to A$ by

$$(f|\eta)(g) = f(g\eta^{-1})\eta_p.$$

Note that using this definition we have

$$\mathcal{L}(U,A) = \{f : D^\times \backslash D_f^\times \to A \,:\, f|\eta = f \text{ for all } \eta \in U\}.$$

Now say η is as above. We define $A^{[\eta_p]}$ to be the abelian group A equipped with the action, denoted $\cdot$, of $\eta_p^{-1}\mathbf{M}_\alpha\eta_p \cap \mathbf{M}_\alpha$, defined by $a \cdot (\eta_p^{-1}\gamma\eta_p) := a\gamma$. The map $f \mapsto f|\eta$ is easily shown to be an isomorphism $\mathcal{L}(U,A) \to \mathcal{L}(\eta^{-1}U\eta, A^{[\eta_p]})$, which we shall denote $|\eta$. If the map $a \mapsto a\eta_p$ induces an isomorphism of abelian groups $A \to A$, then one can think of this map as an isomorphism $A^{[\eta_p]} \to A$ and in this case the map $|\eta$ sends $\mathcal{L}(U,A)$ to $\mathcal{L}(\eta^{-1}U\eta, A)$ isomorphically.

Next we define Hecke operators. Let η be as above, write $U\eta U = \coprod_i U\eta_i$ (a finite union) and define the Hecke operator $[U\eta U] : \mathcal{L}(U,A) \to \mathcal{L}(U,A)$ by $[U\eta U]f = \sum_i f|\eta_i$. Of particular interest are the standard Hecke operators, defined as follows. If l is a prime not dividing δ, then define $\varpi_l \in \mathbf{A}_f$ to be the finite adele which is l at l and 1 at all other finite places. We sometimes think of ϖ_l as an element of $D_f^\times$ via the diagonal embedding. By a mild abuse of notation, we define $\eta_l = \left(\begin{smallmatrix} \varpi_l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to mean the element of $D_f^\times$ which is $\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ at l and is the identity at all other places. We define $T_l = [U\eta_l U]$ and $S_l = [U\varpi_l U]$. An easy computation verifies for example that if $f \in S_k^D(U_1(M))(\varepsilon)$, then for all $l \nmid M$ we have $S_l(f) = \varepsilon(l)l^{k-2}f$. We conclude this section by reminding the reader of the explicit form that the Jacquet-Langlands theorem takes in this setting:

Theorem 2 (Jacquet-Langlands, Shimizu, Arthur). *If $k \geq 3$ then the space $S_k^D(U_1(M))$ is isomorphic to the space $S_k^{\delta\text{-}new}(\Gamma_1(M) \cap \Gamma_0(\delta))$ of classical δ-new forms, and this isomorphism commutes with the action of the standard Hecke operators defined above. If $k = 2$ then $S_k^{\delta\text{-}new}(\Gamma_1(M) \cap \Gamma_0(\delta))$ is isomorphic to the quotient of $S_k^D(U_1(M))$ by the subspace of forms which factor through the norm map, and again the isomorphism commutes with the action of the standard Hecke operators defined above.*

Proof. This is a "concrete" realisation of the Jacquet-Langlands theorem; a good reference for how to deduce it from Theorem 16.1 of [13] is the first few pages of section 5 of [8] (with $t = 1$ in their notation). □

As a consequence we see that if $f \in S_k^D(U_1(M))(\varepsilon)$ is an eigenform then there is a continuous Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(K)$ associated to f satisfying the usual properties.

5 Overconvergent automorphic forms of a fixed weight.

Let K be a complete subfield of $\mathbf{C}_p$ as usual.

Definition. *A real number r is said to be a* radius of convergence *if $r = p^{-n}$ for some integer $n \geq 0$.*

Definition. *If r is a radius of convergence then we let $\mathbf{B}_r$ be the rigid subspace of affine 1-space over K whose $\mathbf{C}_p$-points are*

$$\mathbf{B}_r(\mathbf{C}_p) = \{z \in \mathbf{C}_p : \textit{there exists } y \textit{ in } \mathbf{Z}_p \textit{ such that } |z - y| \leq r\}.$$

If furthermore $r < 1$ then we define $\mathbf{B}_r^\times$ as the rigid subspace of affine 1-space whose $\mathbf{C}_p$-points are

$$\mathbf{B}_r^\times(\mathbf{C}_p) = \{z \in \mathbf{C}_p : \textit{there exists } y \textit{ in } \mathbf{Z}_p^\times \textit{ such that } |z - y| \leq r\}.$$

In fact $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ are just disjoint unions of finitely many closed affinoid discs. One can think of $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ as being a sequence of rigid neighbourhoods of $\mathbf{Z}_p$ and $\mathbf{Z}_p^\times$ respectively in the analytification of affine 1-space over K. If $s < r$ are both radii of convergence then $\mathbf{B}_s \subset \mathbf{B}_r$ and $\mathbf{B}_s^\times \subset \mathbf{B}_r^\times$.

We say that a map between K-Banach spaces is *compact* (some authors use the term "completely continuous") if it is a limit of maps having finite rank.

If r is a radius of convergence then let $\mathcal{A}_r$ denote the ring of functions on $\mathbf{B}_r$, equipped with the supremum semi-norm (which is a norm in this case, as $\mathcal{A}_r$ is reduced). A basic result is

Lemma 3. *If $s < r$ are radii of convergence, then the inclusion $\mathbf{B}_s \subset \mathbf{B}_r$ induces a compact map $\mathcal{A}_r \to \mathcal{A}_s$.*

Proof. This is an immediate consequence of the fact that the inclusion $\mathbf{B}_s \subset \mathbf{B}_r$ is inner. Alternatively, one can give a direct proof, as follows: the space $\mathbf{B}_r$ is a finite disjoint union of closed discs, and hence $\mathcal{A}_r$ is a finite direct sum of rings topologically isomorphic to the affinoid $K\langle x\rangle$. The inclusion $\mathbf{B}_s \to \mathbf{B}_r$ induces, on each affinoid corresponding to a component of $\mathbf{B}_r$, a map $K\langle x\rangle \to K\langle y\rangle$ which, if one chooses appropriate parameters on the discs, is equal to the ring homomorphism defined by $x \mapsto (r/s)y$. But this map is compact because for any $\varepsilon > 0$, the image of the basis $\{1, x, x^2, \ldots\}$ of $K\langle x\rangle$ only contains finitely many terms with norm greater than ε. □

Note that we think of $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ not as being abstract rigid spaces but as being explicit rigid subspaces of rigid affine 1-space. Hence these spaces come with with a fixed parameter z, upon which several constructions below will depend.

Let $\mathcal{W}$ denote "weight space" in this setting, that is, the rigid space

$$\mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{G}_m)$$

over K (see Lemma 2). Then the L-points (L any finite extension of K) of $\mathcal{W}$ are the continuous group homomorphisms $\kappa : \mathbf{Z}_p^\times \to L^\times$. We refer to points of $\mathcal{W}$ as weights, or as weight-characters. Points of $\mathcal{W}$ corresponding to weight-characters of the form $\kappa(x) = x^k \varepsilon_p(x)$, where k is an integer and ε_p is a finite order character, will be called classical weight-characters.

We have seen that $\mathcal{W}$ is a union of finitely many open discs. Say $\kappa \in \mathcal{W}(K)$. Then it is an easy check (use log and exp) that κ extends to a morphism of rigid spaces $\mathbf{B}_r^\times \to \mathbf{G}_m^{\mathrm{an}}$ for some radius of convergence $r < 1$, (and hence to a morphism $\mathbf{B}_s^\times \to \mathbf{G}_{m,K}$ for any radius of convergence $s < r$). For example, one can take $r = p^{-1}$ if $\kappa(x) = x^k$ for some $k \in \mathbf{Z}$, and $r = p^{-n}$ if κ is of finite order and factors through $(\mathbf{Z}/p^n\mathbf{Z})^\times$.

If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_\alpha$ then the map $z \mapsto cz + d$ defines a morphism $\mathbf{B}_r \to \mathbf{B}_{rp^{-\alpha}}^\times$, and the map $z \mapsto \frac{az+b}{cz+d}$ defines a morphism $\mathbf{B}_r \to \mathbf{B}_{r|\Delta|}$, where $\Delta = ad - bc$. We frequently use the weaker statement that $z \mapsto \frac{az+b}{cz+d}$ defines a map $\mathbf{B}_r \to \mathbf{B}_r$. Now say $\kappa \in \mathcal{W}(K)$ and r is a radius of convergence.

Definition. *We say an integer $\alpha \geq 1$ is* good *for (κ, r) if $\kappa : \mathbf{Z}_p^\times \to K^\times$ extends to a morphism of rigid spaces $\mathbf{B}_{rp^{-\alpha}}^\times \to \mathbf{G}_m^{an}$.*

For any κ and r, a good integer α will exist, by the remarks above. Of course, given α which is good for (κ, r), the extension of κ to $\mathbf{B}_{rp^{-\alpha}}^\times$ is unique. Finally, if α is good for (κ, r) and $\beta > \alpha$ then β is also good for (κ, r) and the extension of κ to $\mathbf{B}_{rp^{-\beta}}^\times$ is just the restriction of the extension of κ to $\mathbf{B}_{rp^{-\alpha}}$.

Now choose κ and r as above, and let α be any integer which is good for (κ, r). Let κ also denote the extension of κ to $\mathbf{B}_{rp^{-\alpha}}^{\times}$.

Definition. *Let $\mathcal{A}_{\kappa,r}$ be the ring $\mathcal{A}_r$ of rigid-analytic functions on $\mathbf{B}_r$, equipped with the following right action of $\mathbf{M}_\alpha$: for $h \in \mathcal{A}_{\kappa,r}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_\alpha$,*

$$(h|\gamma)(z) = \frac{\kappa(cz+d)}{(cz+d)^2} h\left(\frac{az+b}{cz+d}\right).$$

Note that α has to be good for (κ, r) for this definition to make sense. Note also that the simplest example of these rings is just $\mathcal{A}_{\kappa,1}$ which is the ring $K\langle z\rangle$ of power series $\sum a_n z^n$ such that $a_n \to 0$ as $n \to \infty$.

We say that a map f between Banach spaces is *norm-decreasing* if $|f(x)| \leq |x|$ for all x. We have equipped $\mathcal{A}_r$ with the structure of a Banach space and we think of $\mathcal{A}_{\kappa,r}$ also as having this Banach space structure. The map $\mathcal{A}_{\kappa,r} \to \mathcal{A}_{\kappa,r}$ defined by $h \mapsto h|\gamma$ is norm-decreasing: this is an immediate consequence of the fact that $\kappa(cz+d)$ and $cz+d$ both have norm 1. If $|\det(\gamma)| < 1$ then one can say much more. In this case, the endomorphism $f \mapsto f|\gamma$ of $\mathcal{A}_{\kappa,r}$ is compact, as it factors as $\mathcal{A}_{\kappa,r} \xrightarrow{\text{res}} \mathcal{A}_{\kappa,r|\Delta|} \xrightarrow{|\gamma} \mathcal{A}_{\kappa,r}$, where $\Delta = \det(\gamma)$, and the restriction map $\mathcal{A}_{\kappa,r} \to \mathcal{A}_{\kappa,r|\Delta|}$ is compact by Lemma 3.

On the other hand if $|\det(\gamma)| = 1$ then γ has an inverse in $\mathbf{M}_\alpha$, and so the induced endomorphism of $\mathcal{A}_{\kappa,r}$ is norm-decreasing with a norm-decreasing inverse and is hence an isometry.

For κ a weight and $r = p^{-n}$, choose $\alpha \geq 1$ which is good for (κ, r), and let U be a compact open subgroup of $D_f^\times$ of wild level $\geq p^\alpha$. The key definition is the following:

Definition. *The space of r-overconvergent automorphic forms of weight-character κ and level U is the space $\mathbf{S}_\kappa^D(U; r) := \mathcal{L}(U, \mathcal{A}_{\kappa,r})$.*

We now make some elementary observations about this space. Firstly, note that this space is independent of the choice of α, in the sense that if β is also good for (κ, r) and U also has wild level $\geq p^\beta$, then the two induced actions of U on $\mathcal{A}_{\kappa,r}$ are the same and hence the definition is independent of the choice of α. Next note that using the "explicit" description of equation (1) in section 4, one sees that $\mathbf{S}_\kappa^D(U; r)$ is naturally a Banach space over K (note that the groups Γ_i are finite and K has characteristic 0 and hence the subspace of $\mathcal{A}_{\kappa,r}$ where Γ_i acts trivially is a closed direct summand). In fact one can write down an explicit norm on $\mathbf{S}_\kappa^D(U; r)$, useful for computational purposes—if $f \in \mathbf{S}_\kappa^D(U; r)$ then define $|f| = \max_{g \in D_f^\times} |f(g)|$. This max exists, is finite, and is attained, because if $u \in U$ and $h \in \mathcal{A}_{\kappa,r}$ then $\det(u_p) \in \mathbf{Z}_p^\times$, so h and $h|u$ have the same norm, and hence $|f(g)|$ is constant on double cosets $D^\times gU$. One can also view the norm on $\mathbf{S}_\kappa^D(U, r)$ in the following manner: fix once and for all $\{d_1, \ldots, d_n\}$ such that $D_f^\times = \coprod D^\times d_i U$; then there is an induced isomorphism $\mathbf{S}_\kappa^D(U; r) \cong \oplus_i \mathcal{A}_{\kappa,r}^{\Gamma_i}$ as above, and the right hand side is a closed subspace of the affinoid $\oplus_i \mathcal{A}_{\kappa,r}$ (which can be thought of as being equipped with its usual supremum semi-norm).

If $s < r$ is another weight, then the restriction map $\mathcal{A}_{\kappa,r} \to \mathcal{A}_{\kappa,s}$ is compact and norm-decreasing, and hence induces a compact norm-decreasing map $\mathbf{S}^D_\kappa(U;r) \to \mathbf{S}^D_\kappa(U;s)$.

For $\eta \in D_f^\times$ such that $\eta_p \in \mathbf{M}_\alpha$, the endomorphism $[U\eta U]$ of $\mathbf{S}^D_\kappa(U;r)$ is continuous and norm-decreasing. Moreover if also η_p has determinant Δ a non-unit, then $[U\eta U]$ factors as

$$[U\eta U] : \mathbf{S}^D_\kappa(U;r) \xrightarrow{\text{res}} \mathbf{S}^D_\kappa(U;r|\Delta|) \to \mathbf{S}^D_k(U;r)$$

and hence is norm-decreasing and compact, as the restriction morphism is compact. Recall that a compact map has a characteristic power series, or a Fredholm determinant, which is the natural generalisation of the characteristic polynomial of a finite rank map: see section 5 of [16] for a beautifully-written introduction to these notions. The fact that $[U\eta U]$ is compact and norm-decreasing means that its characteristic power series is in $\mathcal{O}_K\{\{T\}\}$, that is, in the space of power series that converge on the whole of affine 1-space. The case we are most interested in is the Hecke operator $U_p := [U\eta_p U]$ where we recall $\eta_p = \left(\begin{smallmatrix} \varpi_p & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

If κ is a classical weight-character, that is, $\kappa(x) = x^k \varepsilon_p(x)$ where k is an integer and ε_p is a finite order character, then choose $\alpha \geq 1$ minimal such that ε_p factors through $(\mathbf{Z}/p^\alpha\mathbf{Z})^\times$. We have a natural inclusion $L_{k,\varepsilon_p} \to \mathcal{A}_{\kappa,1}$ of $\mathbf{M}_\alpha$-modules. This induces a map from classical automorphic forms (with p-adic coefficients) to 1-overconvergent automorphic forms. More precisely, if we fix a level U_0 prime to p and any $n \geq \alpha$ then we get a continuous embedding from $S^D_k(U_0 \cap U_1(p^n))(\varepsilon_p) = \mathcal{L}(U_0 \cap U_0(p^n), L_{k,\varepsilon_p})$ to $\mathbf{S}_\kappa(U_0 \cap U_1(p^n), 1)$. This is the analogue in this setting of the statement that classical forms are overconvergent. In the next section we will discuss the slightly more subtle question of letting the level drop at p, and prove an analogue of the result that classical forms of level Np^n with trivial character at p are overconvergent of level Np.

6 The operator U_p

We record a few useful facts about the Hecke operator U_p. Let κ be a weight and let $r = p^{-t}$ be a radius of convergence. Choose any $\alpha \geq 1$ which is good for (κ, r) and choose any $n \geq \alpha$. We will be concerned with forms of level $U = U_0 \cap U_1(p^\alpha) \cap U_0(p^n)$, where U_0 is a compact open of level prime to p.

Lemma 4. *1. There exist maps $\alpha_s : \mathbf{S}^D_\kappa(U;s/p) \to \mathbf{S}^D_\kappa(U;s)$, for all radii of convergence s, such that $U_p : \mathbf{S}^D_\kappa(U;r) \to \mathbf{S}^D_\kappa(U;r)$ factors as*

$$\mathbf{S}^D_\kappa(U;r) \xrightarrow{res} \mathbf{S}^D_\kappa(U;r/p) \xrightarrow{\alpha_r} \mathbf{S}^D_\kappa(U;r)$$

and if $r < 1$ it also factors as

$$\mathbf{S}^D_\kappa(U;r) \xrightarrow{\alpha_{pr}} \mathbf{S}^D_\kappa(U;pr) \xrightarrow{res} \mathbf{S}^D_\kappa(U;r).$$

2. U_p is a compact operator on $\mathbf{S}^D_\kappa(U;r)$.

3. *If $s < r$ is another radius of convergence then the characteristic power series of U_p on $\mathbf{S}_\kappa^D(U;s)$ and on $\mathbf{S}_\kappa^D(U;r)$ coincide.*

4. *There is a canonical isomorphism of Hecke modules*

$$\iota : \mathbf{S}_\kappa^D(U;r/p) \cong \mathbf{S}_\kappa^D(U \cap U_0(p^{n+1});r)$$

5. *The characteristic power series of U_p on $\mathbf{S}_\kappa^D(U;r)$ equals the characteristic power series of U_p on $\mathbf{S}_\kappa^D(U_0 \cap U_1(p^\alpha);r)$. That is, "the characteristic power series does not see the higher $U_0(p^n)$ structure".*

Proof. 1. By definition, U_p is the sum of various maps $f \mapsto f|\gamma$ where $\gamma \in \mathbf{M}_\alpha$ has determinant p. Hence the results follow from the fact that $|\gamma : \mathcal{A}_{\kappa,r} \to \mathcal{A}_{\kappa,r}$ factors as $\mathcal{A}_{\kappa,r} \xrightarrow{\text{res}} \mathcal{A}_{\kappa,r/p} \xrightarrow{|\gamma} \mathcal{A}_{\kappa,r}$, and also as $\mathcal{A}_{\kappa,r} \xrightarrow{|\gamma} \mathcal{A}_{\kappa,pr} \xrightarrow{\text{res}} \mathcal{A}_{\kappa,r}$ if $r < 1$. Hence $\alpha_r(f) := \sum_i f|\eta_i$ will do the job, if $U\begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix}U = \coprod_i U\eta_i$.

2. The restriction maps are compact and compactness of U_p follows.

3. Let $\alpha_{ps} : \mathbf{S}_\kappa^D(U;s) \to \mathbf{S}_\kappa^D(U;ps)$ be as in part 1. The characteristic power series of $\alpha_s \circ \text{res}$ and $\text{res} \circ \alpha_s$ are equal, by [16], Corollaire 2 to Proposition 7, and the result follows easily by induction on $-\log_p(s)$.

4. One constructs ι by composing the following sequence of canonical isomorphisms. Firstly define $U^0(p)$ to be the matrices $\gamma \in U_0(1)$ such that γ_p is lower triangular mod p. Then one observes that $\mathbf{S}_\kappa^D(U;r/p) = \mathcal{L}(U, \mathcal{A}_{\kappa,r/p})$ is the subspace of $\mathcal{L}(U \cap U^0(p), \mathcal{A}_{\kappa,r/p})$ where the matrices $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ all act as the identity. This space is naturally isomorphic to the space $\mathcal{L}(U \cap U^0(p), \mathcal{O}(p\mathbf{B}_r))$, where $p\mathbf{B}_r$ is the rigid subspace of $\mathbf{A}_1^{\text{an}}$ whose $\mathbf{C}_p$-points are $\{z : z/p \in \mathbf{B}_r(\mathbf{C}_p)\}$. Note that $p\mathbf{B}_r \subset \mathbf{B}_{r/p}$. Next we observe that the map $h \mapsto h|\begin{pmatrix} \varpi_p & 0 \\ 0 & 1 \end{pmatrix}$ induces an isomorphism $\mathcal{L}(U \cap U^0(p), \mathcal{O}(p\mathbf{B}_r)) \to \mathcal{L}(U \cap U_0(p^{n+1}), \mathcal{O}(p\mathbf{B}_r)^{[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]})$. Finally, one checks that the map $f \mapsto f|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ induces an isomorphism $\mathcal{O}(p\mathbf{B}_r)^{[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]} \to \mathcal{A}_{\kappa,r}$ and hence an isomorphism $\mathcal{L}(U \cap U_0(p^{n+1}), \mathcal{O}(p\mathbf{B}_r)^{[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}) = \mathcal{L}(U \cap U_0(p^{n+1}), \mathcal{A}_{\kappa,r}) = \mathbf{S}_\kappa^D(U \cap U_0(p^{n+1});r)$. If we define ι to be the composite of these isomorphisms then it is elementary to check that ι commutes with the Hecke operators away from p, and an explicit calculation shows that ι also commutes with the Hecke operator U_p. We remark that the way we have normalised things, the operator U_p at level p^n is $\sum_{i=0}^{p-1} |\begin{pmatrix} p & 0 \\ ip^n & 1 \end{pmatrix}$, and in particular it depends on n. This annoyance could have been avoided at the expense of introducing other annoyances.

5. It suffices to prove that the characteristic power series of U_p on $\mathbf{S}_\kappa^D(U_0 \cap U_1(p^\alpha) \cap U_0(p^n); r/p)$ and $\mathbf{S}_\kappa^D(U_0 \cap U_1(p^\alpha) \cap U_0(p^{n+1}); r)$ are equal, if $n \geq \alpha$. Let T denote the composition

$$\mathbf{S}_\kappa^D(U_0 \cap U_1(p^\alpha) \cap U_0(p^{n+1}); r) \xrightarrow{\text{res}} \mathbf{S}_\kappa^D(U_0 \cap U_1(p^\alpha) \cap U_0(p^{n+1}); r/p) \to$$

$$\to \mathbf{S}^D_\kappa(U_0 \cap U_1(p^\alpha) \cap U_0(p^n); r/p),$$

where this latter morphism is given by the trace map $f \mapsto \sum_{i=0}^{p-1} f|\left(\begin{smallmatrix} 1 & 0 \\ p^n i & 1 \end{smallmatrix}\right)$. Then it is an elementary exercise to check that T is compact, and that both $\iota \circ T$ and $T \circ \iota$ are equal to U_p on their respective spaces. The result follows by [16], Corollaire 2 to Proposition 7.

□

We can now easily prove analogues of the first two parts of Theorem 1. As before, let U_0 be some compact open of level prime to p, let κ be a weight-character and let r be a radius of convergence. Let α be any positive integer which is good for (κ, r), and let $n \geq \alpha$ be any integer. Set $U = U_0 \cap U_1(p^\alpha) \cap U_0(p^n)$.

Proposition 3.

1. *If $f \in \mathbf{S}_\kappa(U; r)$ then $f|\left(\begin{smallmatrix} \varpi_p^{\alpha-n} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (appropriately interpreted) is an element of $\mathbf{S}_\kappa(U_0 \cap U_1(p^\alpha); r/p^{n-\alpha})$.*
2. *If $f \in \mathbf{S}_\kappa(U; r)$ and $U_p f = \lambda f$ for some non-zero λ then $f \in \mathbf{S}_\kappa(U; s)$ for any radius of convergence $s \geq r$ such that α is good for (κ, s).*

Proof.

1. One continually applies the inverse of the map ι defined in part 4 of Lemma 4.
2. By part 1 of Lemma 4, $U_p f \in \mathbf{S}_\kappa(U; pr)$ and hence $f \in \mathbf{S}_\kappa(U; pr)$. So we are finished by induction on $-\log_p(r)$.

□

Note that the map $f \mapsto f|\left(\begin{smallmatrix} \varpi_p^{n-\alpha} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the inverse of the map in part 1 of this proposition, is in fact the natural inclusion $\mathbf{S}_\kappa(U_0 \cap U_1(p^\alpha); r/p^{n-\alpha}) \to \mathbf{S}_\kappa(U; r)$, in the sense that it is the one which commutes with the Hecke operator U_p. The fact that it is this morphism which is natural, rather than the identity map, is a mildly annoying consequence of the way we have set things up—more specifically, it is because the matrices defining U_p depend on the level that we are working with.

7 Classical and overconvergent forms.

We have just seen how classical forms embed into overconvergent ones. Now we show how to pull them out. Let $\kappa = (k, \varepsilon_p)$ be a classical weight-character, where $k \geq 1$, and as usual choose $\alpha \geq 1$ minimal such that ε_p factors through $(\mathbf{Z}/p^\alpha\mathbf{Z})^\times$. Then $\mathbf{M}_\alpha$ acts on L_{k,ε_p}. Let U_0 be a compact open in $D_f^\times$ of level prime to p, and let $U = U_0 \cap U_0(p^n)$ for any $n \geq \alpha$. Let κ' be the character $(2-k, \varepsilon_p)$. We define a map

$$\theta^{1-k} : \mathbf{S}^D_\kappa(U, 1) \to \mathbf{S}^D_{\kappa'}(U, 1)$$

by

$$\left(\theta^{1-k}(f)\right)(g) = (|\nu(g)|\det(g_p))^{1-k}\,\frac{d^{k-1}f(g)}{dz^{k-1}}.$$

Here ν is the norm map $D_f^\times \to \mathbf{A}_f^\times$, and $|.|$ is the usual absolute value map $\mathbf{A}_f \to \mathbf{Q}$. Note that the superscript $1-k$ is notational and merely there to indicate that θ^{1-k} looks like the $(1-k)$th power of some kind of Tate twisting operator.

One has to verify that the map θ^{1-k} is well-defined, which boils down to checking that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_\alpha$ and F a power series in z, we have the identity

$$\left(\frac{d^{k-1}}{dz^{k-1}}\right)\left((cz+d)^{k-2}F(\frac{az+b}{cz+d})\right) = (ad-bc)^{k-1}(cz+d)^{-k}\left(\frac{d^{k-1}F}{dz^{k-1}}\right)(\frac{az+b}{cz+d}).$$

This identity is trivial for $k=1$ and the general case is easily established by induction on k. Next one can analyse the relationship between θ^{1-k} and Hecke operators. Again it is elementary to check that if $f \in \mathbf{S}_\kappa^D(U,1)$ and $\eta \in D_f^\times$ with $\eta_p \in \mathbf{M}_\alpha$, then $(\theta^{1-k}f)|\eta = |\nu(\eta)|^{k-1}\theta^{1-k}(f|\eta)$ and hence that $[U\eta U]\theta^{1-k} = |\nu(\eta)|^{k-1}\theta^{1-k}[U\eta U]$. Applying this with $\eta = \eta_l$ one sees that $T_l\theta^{1-k} = l^{1-k}\theta^{1-k}T_l$ and hence that if $f \in \mathbf{S}_\kappa^D(U,1)$ is an eigenform for T_l with eigenvalue a_l then $\theta^{1-k}f$ is an eigenform for T_l with eigenvalue $a_l l^{1-k}$. We now prove the analogue of the third and fourth results of Theorem 1 in this setting.

Proposition 4. *The kernel of θ^{1-k} is precisely the classical forms*

$$\mathbf{S}_k^D(U_0 \cap U_1(p^n))(\varepsilon_p).$$

Let $0 \neq f \in \mathbf{S}_\kappa^D(U;r)$ be an eigenvector for U_p with non-zero eigenvalue λ. Then $f \in \mathbf{S}_\kappa^D(U;1)$. Moreover, if $v_p(\lambda) < k-1$ then f is classical, and if $v_p(\lambda) > k-1$ then f is not classical.

Proof. The kernel of θ^{1-k} is the functions $f \in \mathbf{S}_\kappa^D(U;1)$ whose image is contained within the space of polynomials of degree at most $k-2$, which is precisely the space of classical forms $\mathcal{L}(U, L_{k,\varepsilon_p}) = S_k^D(U_0 \cap U_1(p^n))(\varepsilon_p)$. Next, let $0 \neq f \in \mathbf{S}_\kappa^D(U;r)$ be an eigenvector for U_p with non-zero eigenvalue λ. By the proposition in the previous section, $f \in \mathbf{S}_\kappa^D(U,1)$.

If f is classical then one can easily deduce from the classical theory (see for example Theorem 4.6.17 of [15], the fact that λ is an algebraic integer, and the Jacquet-Langlands theorem) that $v_p(\lambda) \leq k-1$. On the other hand, if $v_p(\lambda) < k-1$ then $\theta^{1-k}f$ is in $\mathbf{S}_{\kappa'}^D(U;1)$ and if it is non-zero then it is an eigenvector for U_p with eigenvalue λ/p^{k-1}, which has negative valuation. On the other hand, U_p is an operator with norm at most 1, and hence $\theta^{1-k}f = 0$. Hence f is classical. □

In the classical theory, one can find both classical and non-classical forms with $v_p(\lambda) = k-1$, and it would be an interesting computational exercise to search for examples of this phenomenon in this situation.

We finish by remarking that if we had defined our spaces of overconvergent automorphic forms as $\mathcal{L}(U, (\mathcal{A}_{\kappa,r})^*)$, using the dual of $\mathcal{A}_{\kappa,r}$, then one would see the theta operator going in the other direction, which would be more analogous to the classical theory. On the other hand, using this convention gives a surjection from overconvergent forms to classical ones, rather than an injection from classical forms to overconvergent ones. Using the duals would give a theory more analogous to the one set up by Stevens in the classical case.

8 Families

Let R be a reduced affinoid over K. Then R is a Banach algebra in the sense of section A1 of [6]. Let A be a Banach module over R. Briefly, the key points are that R is a ring complete with respect to a non-trivial ultrametric norm and A is a complete normed R-module satisfying various natural axioms such as $|ra| \leq |r||a|$ for $r \in R$ and $a \in A$. In fact the only cases of interest to us in this paper are when R is an affinoid disc over K. Assume furthermore that we have an R-linear action of $\mathbf{M}_\alpha$ on A, and that all elements of $\mathbf{M}_\alpha$ act continuously. Then for $U \subset D_f^\times$ open and compact and of wild level $\geq p^\alpha$, we see that $\mathcal{L}(U, A)$ also inherits the structure of a Banach module over R. Let us for simplicity assume in this section that all the finite groups Γ_i are trivial.

We now specialise to the case that we are interested in. Let R be the ring of functions on an affinoid disk V defined over K in $\mathcal{W}$. Define $\mathcal{A}_{V,r}$ to be the ring of functions on the rigid space $V \times_K \mathbf{B}_r$. We say that an integer $\alpha \geq 1$ is *good* for the pair (V, r) if for all $\kappa \in V(\mathbf{C}_p)$, the map $\kappa : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ extends to a map $\mathbf{B}_r^\times \to \mathbf{G}_m^{\mathrm{an}}$. Because V is affinoid, one can check that good integers α do exist. Define a right action of $\mathbf{M}_\alpha$ on $\mathbf{A}_{V,r}$ by, for $h \in \mathbf{A}_{V,r}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_\alpha$,

$$(h|\gamma)(\kappa, z) = \frac{\kappa(cz+d)}{(cz+d)^2} h\left(\kappa, \frac{az+b}{cz+d}\right).$$

It is easily verified that $\mathbf{M}_\alpha$ acts by continuous R-linear maps. If U is a compact open of wild level $\geq p^\alpha$ then we define $\mathbf{S}_V^D(U; r) := \mathcal{L}(U, \mathcal{A}_{V,r})$. This is a Banach module over R and it enjoys various base change properties. For example, if $V_1 \subseteq V_2$ are affinoids in $\mathcal{W}$, with associated algebras R_1 and R_2, then $\mathbf{S}_{V_2}^D(U; r)\widehat{\otimes}_{R_2} R_1 = \mathbf{S}_{V_1}^D(U; r)$, and if $\kappa \in V(K)$ inducing a map $R \to K$ then $\mathbf{S}_V^D(U; r) \otimes_R K = \mathbf{S}_\kappa^D(U; r)$. The same formalism as we have seen already in the case of fixed weight shows that we have continuous Banach R-module endomorphisms T_l of $\mathbf{S}_V^D(U; r)$ for l any prime not dividing δ, and furthermore that U_p is a compact morphism. In particular U_p on $\mathbf{S}_V^D(U, r)$ has a characteristic power series in $R\{\{T\}\}$ whose restriction to a weight κ is the characteristic power series of U_p on forms of weight κ.

The reader who has read the construction of the classical eigencurve in [7] will now see that we have all the ingredients to construct the analogous object in this case. The reason we have restricted to the case where the Γ_i are all trivial is to ensure that all the Banach modules $\mathbf{S}_V^D(U, r)$ are orthonormizable, although in future work we shall show that this assumption is unnecessary. We point out here that we only define the "D" eigencurve in this setting, although no doubt one can also construct a "C" eigencurve by mimicking the construction in [7].

By the argument at the beginning of Chapter 7 of [7], for every Hecke operator α we can construct a spectral curve Z_α representing the spectrum of the compact operator αU_p. All of Coleman's Riesz theory applies in this setting, and the construction of the curve D_α in Section 7.2 of [7] goes through word for word in our setting. Similarly the arguments of Section 7.3 go over unchanged, to give us an eigencurve D. Once the existence of D is established, the usual corollaries, such as local constancy of dimension of spaces of overconvergent forms of a given slope follow, although of course one can prove these corollaries directly just by an analysis of the spaces $\mathbf{S}_V^D(U,r)$. Finally, the fact that forms of small slope are classical enables one to deduce that the dimensions of spaces of classical forms of a given slope are also locally constant.

We remark that Dan Jacobs in his thesis ([12]) has make some explicit computations which indicate that, just as in the classical case, these new eigencurves exhibit some surprising and unexplained regularity in their structure.

We end with a remark about what is lacking in this theory. One can prove slightly less about these eigencurves than the classical eigencurves, because the arguments in [7] that rely on the existence of q-expansions seem to have no analogue in this setting. The fact that points on the eigencurve correspond to overconvergent eigenforms follows from some elementary commutative algebra, but on the other hand, the lack of a natural pairing between spaces of forms and Hecke algebras in this setting means that the author cannot currently prove that the rank of the Hecke algebra acting on a space of overconvergent forms of some fixed slope is as big as the rank of the space of the forms. We hope to resolve this problem in the future.

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