Integral models of certain Shimura curves

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§0. Introduction.

Much of the theory of the mod p reduction of modular curves is now well-understood and well-documented (for example in [KM]). Although a lot is also known to the experts in the more general setting of curves attached to arbitrary indefinite quaternion algebras over $\mathbb{Q}$, less seems to have been written down. In particular, the analogues of the results of Deligne and Rapoport on the bad reduction of $X_0(p)$ and $X_1(p)$ seem not to be in the literature. Over totally real fields other than $\mathbb{Q}$, one has the article [C] of Carayol, but the case of $\mathbb{Q}$ is excluded in [C]. Theorem 4.7 and Theorem 4.10 of this paper are the analogues of the results of Deligne and Rapoport in this more general setting, and the results show a strong analogy with the modular curve case, as expected.

§1. False elliptic curves.

Fix an indefinite non-split quaternion algebra D over $\mathbb{Q}$, and let $d = \mathrm{disc}(D)$. By definition, d is the product of the (finitely many) primes p for which $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is not isomorphic to $\mathrm{M}_2(\mathbb{Q}_p)$. Fix once and for all a maximal order $\mathcal{O}_D$ of D.

Let $R = \varprojlim(\mathbb{Z}/M\mathbb{Z})$, where the limit is over all positive integers M prime to d. Fix once and for all an isomorphism $\kappa : \mathcal{O}_D \otimes_{\mathbb{Z}} R \to \mathrm{M}_2(R)$, and note that κ induces an isomorphism $\mathcal{O}_D \otimes_{\mathbb{Z}} T \cong \mathrm{M}_2(T)$ for all quotients T of R. Write $\mathcal{O}_{D,f}$ for $\mathcal{O}_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Then κ induces a natural map $\mathcal{O}_{D,f} \to \mathrm{M}_2(\mathbb{Z}/N\mathbb{Z})$ for any positive integer N prime to d, by the composite of the maps $\mathcal{O}_{D,f} \to \mathcal{O}_D \otimes_{\mathbb{Z}} R \xrightarrow{\kappa} \mathrm{M}_2(R) \twoheadrightarrow \mathrm{M}_2(\mathbb{Z}/N\mathbb{Z})$. Finally, fix once and for all an isomorphism $\kappa_\infty : D \otimes \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$.

Let S be a scheme on which d is invertible. A *false elliptic curve* over S is a pair $(A/S, i)$ where A/S is an abelian surface (that is, A is an abelian scheme over S of relative dimension two), and $i : \mathcal{O}_D \hookrightarrow \mathrm{End}_S(A)$ is an injective ring homomorphism (where throughout this paper all ring homomorphisms are assumed to be identity-preserving). We will frequently abuse notation and talk of "the false elliptic curve A/S", or even just "the false elliptic curve A". Strictly speaking we should really refer to "false elliptic curves for D", but because we shall never vary our choice of D it will hopefully cause no confusion if we omit it. Also, we shall only ever refer to false elliptic curves over $\mathbb{Z}[1/d]$-schemes. Some references for false elliptic curves are [B], [BC], Chapter 4 of [DT] and Chapter 1 of [R]. If A/S is a false elliptic curve, and T is an S-scheme, the pullback $A_T = A \times_S T$ can be naturally given the structure of a false elliptic curve over T.

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Let $x \mapsto x^\iota$ denote the canonical involution of D. Fix $t \in \mathcal{O}_D$ with $t^2 = -d$. Such t exist because $\mathbb{Q}(\sqrt{-d})$ splits D. Define another involution $*$ on D by $x^* = t^{-1}x^\iota t$. A local calculation shows that $\mathcal{O}_D$ is stable under $*$. We now observe that if $(A/S, i)$ is a false elliptic curve, then there is a unique principal polarisation on A such that if $\overline{s}$ is a geometric point of S, then the corresponding Rosati involution on $\mathrm{End}(A_{\overline{s}})$ induces $*$ on $\mathcal{O}_D$. This is proved for S the spectrum of an algebraically closed field of characteristic 0 in [BC] III.1.5. Moreover, the proof extends to characteristic p not dividing d, after one observes that if A/k is a false elliptic curve over an algebraically closed field of characteristic p, then the p-divisible group $A[p^\infty]$ is of the form $G \times G$, with $\mathcal{O}_D$ acting via the natural action of $\mathrm{M}_2(\mathbb{Z}_p)$. Finally, we deduce the result for a general base $S/\mathrm{Spec}(\mathbb{Z}[1/d])$ by the Proposition in Section 11 of [B]. We conclude that once we have fixed t, any false elliptic curve admits a canonical principal polarisation.

If $(A/S, i)$ and $(B/S, j)$ are false elliptic curves, then by an *isogeny* $\pi : A \to B$ of false elliptic curves we mean an isogeny $\pi : A \to B$ over S in the usual sense, such that $j(x) \circ \pi = \pi \circ i(x)$ for all $x \in \mathcal{O}_D$. If $\pi : A \to B$ is an isogeny of false elliptic curves, then π induces a dual isogeny $\pi^\vee : B^\vee \to A^\vee$ which, because we have defined principal polarisations on A and B, induces a map $\pi^t : B \to A$. One may check that this map is also an isogeny of false elliptic curves, and we refer to it as the *dual isogeny* to π. The composite $\pi^t \circ \pi : A \to A$ is locally multiplication by an integer d, and if this integer is constant on S we call it the *false degree* of π. There is a canonical perfect pairing

$$\langle , \rangle_\pi : \ker(\pi) \times \ker(\pi^t) \to \mathbb{G}_m.$$

In fact, there are two canonical perfect pairings, one being the inverse of the other. To fix our ideas, we shall use the opposite of the definition in [O]. Lemma 1.1 of [O] then shows us that if $\delta \in D$, $P \in (\ker \pi)(S)$ and $Q \in (\ker \pi^t)(S)$, then $\langle i(\delta)P, Q\rangle_\pi = \langle P, j(\delta^*)Q\rangle_\pi$ in $\mathbb{G}_m(S)$.

We will now recall how to attach various (naïve) level structures to false elliptic curves, analogous to the level structures that one can define on elliptic curves. Let M be a positive integer prime to d, and write $(\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})_S$ for the constant group scheme over S associated to the finite group $\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z}$. Note that left multiplication induces a left action of $\mathcal{O}_D$ on $(\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})_S$.

Definition 1.1. A *naïve full level M structure* on a false elliptic curve $(A/S, i)$ is an isomorphism

$$\alpha : (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})_S \xrightarrow{\cong} A[M]$$

of schemes which preserves the left action of $\mathcal{O}_D$.

Note that the existence of a full level M structure on A/S implies that M is invertible on S.

If $\delta \in \mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z}$ then right multiplication by δ induces a morphism of group schemes

$$r_\delta : (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})_S \to (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})_S$$

which preserves the left action of $\mathcal{O}_D$. Moreover, if $\delta \in (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$ then r_δ is an isomorphism. Hence if α is a full level M structure on a false elliptic curve A, and $\delta \in$

$(\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$, then $\alpha \circ r_\delta$ is also a full level M structure on A. The map $\alpha \mapsto \alpha \circ r_\delta$ defines a left action of $(\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$ on the set of full level M structures on A.

Let $H \subseteq (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$ be a subgroup, let S be a $\mathbb{Z}[1/Md]$-scheme, and let $(A/S, i)$ be a false elliptic curve. We will now define the notion of a naïve level H structure on A. Consider the contravariant functor $F_{A/S,M}$ from S-schemes to sets, sending T/S to the set of full level M structures on A_T. Then $F_{A/S,M}$ is represented by a closed subscheme of $A[M]$, and this representing scheme is an étale $(\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$-torsor. For each T, there is a left action of $(\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$, and hence of H, on $F_{A/S,M}(T)$ as defined above. If $H\backslash F_{A/S,M}(T)$ denotes the orbit space, then define $F_{A/S,H}$ to be the sheafification with respect to the étale topology of the functor $T \mapsto H\backslash F_{A/S,M}(T)$ from S-schemes to sets. Then $F_{A/S,H}$ is represented by a quotient of the S-scheme representing $F_{A/S,M}$ and is again a finite étale covering of S.

Definition 1.2. A *naïve level H structure* on $(A/S, i)$ is an element of $F_{A/S,H}(S)$.

The natural map $\mathcal{O}_{D,f} \to \mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z}$ induces a homomorphism $\mathcal{O}_{D,f}^\times \to (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$. Let $U \subseteq \mathcal{O}_{D,f}^\times$ be the preimage of H under this homomorphism. It is not difficult to check that the definition of a level H structure depends only on U, in the following sense. Let M_1 and M_2 be two positive integers prime to d, with $M_1 \mid M_2$. Let $H_1 \subseteq (\mathcal{O}_D \otimes \mathbb{Z}/M_1\mathbb{Z})^\times$ be a subgroup, and let H_2 be its preimage in $(\mathcal{O}_D \otimes \mathbb{Z}/M_2\mathbb{Z})^\times$ under the natural surjection. Then the notion of an H_1 structure and an H_2 structure coincide for false elliptic curves over $\mathbb{Z}[1/M_2 d]$-schemes. Hence it makes sense to talk about the functor $F_{A/S,U}$ and also of naïve level U structures on a false elliptic curve A/S, for U a compact open subgroup of $\mathcal{O}_{D,f}^\times$.

Finally we record how naïve level H structures may be pushed forward in some cases. Say $(A/S, i)$ and $(B/S, j)$ are false elliptic curves, and $\psi : A \to B$ is an isogeny of false elliptic curves. Say $H \subseteq \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ is a subgroup, and ϕ is a naïve level H structure on A/S. Assume moreover that the false degree of ψ is coprime to M. Then ψ induces an isomorphism $A[M] \to B[M]$ and hence composition with ψ induces an isomorphism of functors $F_{A/S,M} \to F_{B/S,M}$. Furthermore, this isomorphism preserves the action of $\mathcal{O}_D$ and hence induces an isomorphism $F_{A/S,H} \cong F_{B/S,H}$, and hence a bijection ψ_* from the naïve level H structures on A/S and the naïve level H structures on B/S. In particular, $\psi_*\phi$ is a naïve level H structure on B/S.

§2. Shimura curves.

Let M be a positive integer prime to d. Let u_M be the map $u_M : \mathcal{O}_{D,f}^\times \to \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ induced by the map κ of §1.

We now make the following three definitions.

Let $V_0(M)$ denote the preimage of $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z}) | c = 0\}$ under u_M.

Let $V_1(M)$ denote the preimage of $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z}) | c = 0 \text{ and } d = 1\}$ under u_M.

Let V_M be the kernel of the map u_M.

Let U be a compact open subgroup of $\mathcal{O}_{D,f}^\times$ with the following three properties:

(i) $\det(U) = \widehat{\mathbb{Z}}^\times$

(ii) U is maximal at primes dividing d, that is, $U = \prod_{p|d}(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times U^d$, where U^d is the projection of U onto $(\mathcal{O}_D \otimes_{\mathbb{Z}} R)^\times \cong \prod_{p\nmid d}(\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$.
(iii) $U \subseteq V_1(N)$ for some $N \geq 4$ prime to d.

Not all of these assumptions will be necessary at all times, but for simplicity we shall assume them from the outset. Briefly, here are the reasons that these assumptions are here. (i) is convenient because it implies that the complex curve attached to U (see below) is connected. (ii) is much more important, as it ensures that we never consider level structures at primes dividing d, and (iii) is a smallness criterion which we shall need for later representability results. If we relaxed the condition that our objects of study be schemes, and instead allowed them to be algebraic stacks, then condition (iii) would be unnecessary. The author has applications to modular forms in mind, however, and the restrction to schemes will suffice for these purposes.

Define $U_\infty \subseteq (D \otimes \mathbb{R})^\times$ to be the stabiliser of $i = \sqrt{-1}$ under the natural action of $(D \otimes \mathbb{R})^\times \cong \mathrm{GL}_2(\mathbb{R})$ on $\mathcal{H}^\pm$, the union of the upper and lower half planes. Then there is a complex curve $X^D(U)_{\mathbb{C}}$ whose $\mathbb{C}$-valued points are naturally in bijection with the set

$$D^\times \backslash (D \otimes_{\mathbb{Q}} \mathbb{A})^\times / U.U_\infty,$$

where $\mathbb{A}$ denotes the ring of adèles over $\mathbb{Q}$. Our aim is to study certain integral models of these curves. To do this is will be convenient to introduce the language of algebraic stacks, because this is the language of many of the theorems in the literature, although ultimately we shall prove that all the stacks we consider are in fact schemes. One reference for algebraic stacks is [CF], and others are mentioned therein.

Let U be a compact open subgroup of $\mathcal{O}_{D,f}^\times$, satisfying (i)–(iii) above. Let M_U denote the smallest positive integer M prime to d such that $V_M \subseteq U$. Let $X^D(U)$ be the following category. An object of $X^D(U)$ is a false elliptic curve $(A/S/\mathbb{Z}[1/M_U d], i_A)$ equipped with a level U structure $\alpha \in F_{A/S,U}(S)$, and a morphism from $((A/S, i_A), \alpha)$ to $((B/T, i_B), \beta)$ in this category is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ S & \longrightarrow & T \end{array}$$

such that

- The induced morphism $A \to B_S = B \times_T S$ is an isomorphism,
- The induced map $a : \mathrm{End}_T(B) \to \mathrm{End}_S(A)$ satisfies $i_A = a \circ i_B$.
- α is the image of β under the natural map $F_{B/T,U}(T) \to F_{B_S/S,U}(S) = F_{A/S,U}(S)$.

There is a morphism from $X^D(U)$ to the category of $\mathbb{Z}[1/M_U d]$-schemes, sending $((A/S, i_A), \alpha)$ to S.

Theorem 2.1. $X^D(U)$ is an algebraic stack over $\mathbb{Z}[1/M_U d]$. Moreover, the morphism $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ is proper and smooth of relative dimension 1.

Proof. This is proved in [B], section 14. The representability result is deduced from a theorem of M. Artin.

□

We will now use the smallness criterion (iii) on U to show that $X^D(U)$ is in fact the algebraic stack associated to a scheme. For this we shall need

Lemma 2.2. Let $N \geq 4$ be an integer prime to d, and let k be an algebraically closed field of either characteristic 0 or of characteristic p prime to Nd. Let $(A/k, i)$ be a false elliptic curve over k, and let α be a $V_1(N)$-structure on A/k. Then the only automorphism of $(A/k, i)$ which fixes α is the identity.

Proof. Because k is an algebraically closed field, the $V_1(N)$-structure α can be thought of as an orbit of full level N structures under the action of the group $H = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\}$. Let $\beta : (\mathcal{O}_D \otimes \mathbb{Z}/N\mathbb{Z})_k \to A[N]$ be an element of this orbit. We must show that if θ is an automorphism of $(A/k, i)$ such that $\theta\beta = \beta r_\delta$ for some $\delta \in H$, then θ is the identity. So let θ be such an automorphism.

Let $\psi = \theta - 1$. Then ψ is an endomorphism of $(A/k, i)$. Let $G = \ker(\psi)$. Now G is a group scheme of finite type over k, with an action of $\mathcal{O}_D$. This action induces an action of $\mathcal{O}_D$ on G^0, the identity component of G, and hence on $H = (G^0)^{\mathrm{red}}$, the reduced scheme associated to G^0. We see that G/H is a finite group scheme over k, and that H is an abelian variety. Moreover, we have an action of $\mathcal{O}_D$ on H.

If H were 1-dimensional, then it would be an elliptic curve and hence there would be a ring homomorphism (sending 1 to 1) from $\mathcal{O}_D$ to $\mathrm{End}_k(H)$. Tensoring up by $\mathbb{Q}$, we find a contradiction, because are no maps (sending 1 to 1) from D to $\mathbb{Q}$, to an imaginary quadratic extension of $\mathbb{Q}$, or to a definite quaternion algebra over $\mathbb{Q}$.

Hence H is either zero or 2-dimensional. If H is 2-dimensional, then $H = G$ and $\psi = 0$, which implies $\theta = 1$ and we are finished. So assume for a contradiction that $H = 0$. Then G is finite and ψ is an isogeny. If ψ^t is the transpose of this isogeny, then the composite $\psi^t \circ \psi$ is multiplication by a positive integer n, the false degree of ψ. Moreover, θ is an automorphism and hence $\theta^t \circ \theta$ is the identity. We have

$$\begin{aligned} n = \psi^t\psi &= (\theta - 1)^t(\theta - 1) \\ &= \theta^t\theta - \theta^t - \theta + 1 \\ &= 2 - (\theta^t + \theta). \end{aligned}$$

Hence $(\theta^t + \theta)$ is multiplication by an integer, namely $m = 2 - n$. We deduce that θ satisfies the equation $X^2 - mX + 1 = 0$ in $\mathrm{End}_k(A)$.

Now θ is an automorphism of the abelian variety A/k, commuting with the action of $\mathcal{O}_D$. One can check that this implies that it preserves the principal polarisation defined in §1. By [M] proposition 17.5, this implies that θ has finite order in $\mathrm{End}_k(A)$. Say θ has order f. Then set $C = \mathbb{Z}[\theta] \subseteq \mathrm{End}_k(A)$, the ring generated by θ in $\mathrm{End}_k(A)$. Now $C \otimes \mathbb{Q}$ must be a quotient of $\mathbb{Q}[T]/(T^f - 1)$, and the fact that $\theta^2 - m\theta + 1 = 0$ and that $C \neq 0$ implies that $T^2 - mT + 1$ and $T^f - 1$ must have a common factor in $\mathbb{Q}[T]$. Hence one of the roots of $T^2 - mT + 1 = 0$ is a root of unity and hence $|m| \leq 2$. But by definition, $m = 2 - n$ and hence $1 \leq n \leq 4$ because n is a positive integer.

The k-valued point $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of $(\mathcal{O}_D \otimes \mathbb{Z}/N\mathbb{Z})_k$ is mapped by β to a k-valued point Q of A. We observe that $\beta = \theta^t\theta\beta = \theta^t\beta r_\delta$ and hence that $\theta^t\beta = \beta r_{\delta^{-1}}$. Hence $m\beta = \theta\beta + \theta^t\beta = \beta(r_\delta + r_{\delta^{-1}}) = \beta(r_{\delta+\delta^{-1}})$. Applying these maps to P, we deduce that $mQ = 2Q$ and hence $nQ = 0 \in A[N](k)$. Because β is an isomorphism we deduce that $nP = 0$ in $M_2(\mathbb{Z}/N\mathbb{Z})$ and hence that N divides n. If $N \geq 5$ then this is the desired contradiction, so we can assume henceforth that $N = 4$. Because N divides n, we must have $n = 4$ and $m = -2$, so $\theta^2 + 2\theta + 1 = 0$. Hence $\theta + 1$ is certainly not an isogeny, as its square is zero. We deduce that $\theta + 1$ must be zero, and so $\theta = -1$. So $\beta = -\beta r_\delta$, and applying these maps to P we deduce that $Q = -Q$ and hence that Q is a point of order 2, a contradiction because P is of order 4.

□

Corollary 2.3. If $U \subseteq \mathcal{O}_{D,f}^\times$ is open and compact, and satisfies properties (i)–(iii) above, then $X^D(U)$ is the stack associated to a regular scheme, and the morphism $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ is projective.

Proof. Firstly we show that $X^D(U)$ is the stack associated to an algebraic space. To do this, it suffices by Example 4.9 of [DM] to show that the objects of the category $X^D(U)$ have no non-trivial automorphisms. Assume for a contradiction that A/S is an abelian scheme with a level U structure α, and there is an automorphism $\theta \neq 1$ preserving α. By [DR] II 1.14 there is a geometric point $k \to S$ of S such that $\theta_k : A_k \to A_k$ is not the identity. But this contradicts Lemma 2.2, because $U \subseteq V_1(N)$ for some $N \geq 4$. Hence $X^D(U)$ is an algebraic space.

Now $\mathbb{Z}[1/M_U d]$ is regular, and the morphism $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ is smooth of relative dimension 1, and proper. It follows that $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ is quasi-projective, and hence by [Kn] (II.7.6) we see that $X^D(U)$ is a scheme. Finally, the map $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ is quasiprojective and proper, hence projective by [Kn] II.7.8.

□

Finally, we record two useful propositions.

Proposition 2.4. If U satisfies (i)–(iii) above, then the fibres of the morphism $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ are geometrically irreducible.

Proof. By the theory over $\mathbb{C}$ (see [Gi]), the generic fibre of $X^D(U)$ is geometrically connected. By Zariski's connectedness theorem, all the fibres of $X^D(U) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d])$ are geometrically connected. By [B] §16, Corollary, they are geometrically unibranch and hence geometrically irreducible.

□

Proposition 2.5. If $U_1 \subseteq U_2$ both satisfy (i)–(iii) above, then the canonical forgetful morphism $X^D(U_1) \to X^D(U_2)$ over $\mathrm{Spec}(\mathbb{Z}[1/M_{U_1} d])$ is étale.

Proof. Clear.

§3. New structures at l.

We now consider more carefully the l-power torsion in false elliptic curves, where throughout this section l is a prime not dividing d. Let A/S be a false elliptic curve over a $\mathbb{Z}[1/d]$-scheme S. Then the l-divisible group $A[l^\infty]$ attached to A inherits an action of $\mathcal{O}_D$ and hence of $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_l$, which via the map κ of §1 can be identified with $\mathrm{M}_2(\mathbb{Z}_l)$.

If we set $\tilde{e} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_l)$ then $A[l^\infty]$ splits as $\ker(\tilde{e}) \times \ker(1-\tilde{e})$. Moreover, $\ker(\tilde{e})$ and $\ker(1-\tilde{e})$ are isomorphic, because $\tilde{e}$ and $1-\tilde{e}$ are conjugate in $\mathrm{M}_2(\mathbb{Z}_l)$. More specifically, the automorphism of $A[l^\infty]$ induced by the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_l)$ induces an isomorphism $\ker(\tilde{e}) \to \ker(1-\tilde{e})$. Henceforth we shall identify $\ker(1-\tilde{e})$ with $\ker(\tilde{e})$ via this isomorphism.

If $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field, then we can completely classify the possibilities for $\ker(\tilde{e})$, as follows.

• If the characteristic of k is not equal to l, then $A[l^\infty] \cong (\mathbb{Q}_l/\mathbb{Z}_l)^4$ and hence $\ker(\tilde{e}) \cong (\mathbb{Q}_l/\mathbb{Z}_l)^2$.

• If the characteristic of k is equal to l, then there are two possibilities. A is an abelian variety and it is a classical result that the l-divisible group $A[l^\infty]$ is isomorphic to $(\mu_{l^\infty})^r \times (\mathbb{Q}_l/\mathbb{Z}_l)^r \times \Gamma$, where Γ/k is a local-local l-divisible group. Because we have seen that $A[l^\infty] \cong G \times G$ for some l-divisible group G, we deduce that either $r = 2$ and $G = \mu_{l^\infty} \times \mathbb{Q}_l/\mathbb{Z}_l$, or $r = 0$ and G is the unique one-parameter formal Lie group over k of height 2. In particular, in the former case we have $A[l^\infty] \cong (E[l^\infty]^2)$ for E/k any ordinary elliptic curve, and in the latter case $A[l^\infty] \cong (E[l^\infty]^2)$ for E/k any supersingular elliptic curve. Note that in either case there is an elliptic curve E/k with $A[l^\infty] \cong (E[l^\infty]^2)$, a fact which will be very useful to us later, because it will enable us to deduce results about the local rings of the Shimura curves we are studying from facts about the local rings of modular curves.

If A/k is a false elliptic curve over any field of characteristic l, then we have just seen above that the l-divisible group of $A_{\overline{k}}$ is isomorphic to $E[l^\infty]^2$ for $E/\overline{k}$ some elliptic curve. We shall call A/k *ordinary* or *supersingular* depending on whether $E/\overline{k}$ is ordinary or supersingular.

We shall now consider some much more subtle moduli problems. As before, l is a prime not dividing d, and $U \subseteq \mathcal{O}_{D,f}^\times$ is a compact open subgroup with properties (i)–(iii) above. We have seen that $X^D(U)$ is a scheme over $\mathbb{Z}[1/M_U d]$. We shall now "build" on this scheme by adding some structures at l, analogous to structures already considered on elliptic curves.

Definition 3.1. A $\Gamma_0(l)$*-structure* on a false elliptic curve A/S is an isogeny of false elliptic curves $A/S \to B/S$ of false degree l.

Remark. We have imposed no "cyclicity" condition in this definition, for simplicity. We can do this because l is prime. Compare the definitions of a $\Gamma_0(l)$ structure on an elliptic curve in [DR] and in [KM], and also [KM] Corollary 6.8.7.

It is sometimes helpful to have an alternative, equivalent definition of a $\Gamma_0(l)$-structure, which we shall now describe. If A/S is a false elliptic curve, then $A[l]$ is a finite S-group scheme with an action of $\mathcal{O}_D \otimes \mathbb{F}_l$, which via the map κ of §1 can be identified with $\mathrm{M}_2(\mathbb{F}_l)$. If $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{F}_l)$, and we write $G_1 = \ker(e : A[l] \to A[l])$ and $G_2 = \ker(1-e : A[l] \to A[l])$, then $A[l] \cong G_1 \times G_2$.

If $\psi : A \to B$ is an isogeny of false elliptic curves over S, with false degree l, then

$\ker\psi$ is a closed subgroup scheme of $A[l]$, which is finite and locally free of rank l^2, and is fixed by the action of $\mathrm{M}_2(\mathbb{F}_l)$. Hence $\ker\psi \cong K_1 \times K_2 \subseteq G_1 \times G_2$, where K_1 is the kernel of $e : \ker\psi \to \ker\psi$ and K_2 is the kernel of $1-e$ on $\ker\psi$. Note that K_1/S is finite and locally free of rank l, and isomorphic to K_2.

Conversely, if K_1 is an arbitrary finite flat subgroup scheme of G_1 which is locally free of rank l, then setting K_2 to be the subgroup scheme of G_2 which becomes identified with K_1 under the isomorphism of G_1 with G_2 induced by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{F}_l)$, we see that $K_1 \times K_2$ is an $\mathrm{M}_2(\mathbb{F}_l)$-invariant subgroup of $A[l]$ of rank l^2, and $\psi : A \to A/(K_1 \times K_2)$ is an isogeny of abelian surfaces. Moreover, $A/(K_1 \times K_2)$ can be given the structure of a false elliptic curve in such a way that ψ becomes an isogeny of false elliptic curves of false degree l.

We deduce that there are bijections between the $\Gamma_0(l)$-structures on A, the closed $\mathcal{O}_D$-invariant subgroups of $A[l]$ which are locally free of rank l^2, and the closed subgroups K_1 of $\ker(e : A[l] \to A[l])$ which are locally free of rank l.

We now turn to some other structures, analogous to structures already defined on elliptic curves. Recall that $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{F}_l)$ and if A/S is a false elliptic curve then e acts on $A[l]$. Recall also from [KM] that if K is a finite flat group scheme over S which is locally free of rank n, then a point $P \in K(S)$ is called a *generator* of K if the n points $P, 2P, 3P, \ldots, nP$ form a "full set of sections" for K ([KM] 1.8).

Let A/S be a false elliptic curve, where S is a $\mathbb{Z}[1/d]$-scheme as usual. As usual, l is a prime not dividing d.

Definition 3.2. A $\Gamma_1(l)$-structure on A/S is

- $\pi : A \to B$ an isogeny of false elliptic curves, of false degree l,
- $P \in K_1(S)$ a generator of $K_1 = \ker(e : \ker\pi \to \ker\pi)$.

Definition 3.3. A *balanced* $\Gamma_1(l)$*-structure* on $(A/S, i)$ is the following collection of data:

- $\pi : A \to B$ an isogeny of false elliptic curves, of false degree l,
- $\pi^t : B \to A$ the dual isogeny.
- $P \in K_1(S)$ a generator of $K_1 = \ker(e : \ker\pi \to \ker\pi)$,
- $P' \in K_1'(S)$ a generator of $K_1' = \ker(e^* : \ker\pi^t \to \ker\pi^t)$.

Now assume that S is a $\mathbb{Z}[1/d][\zeta_l]$-scheme, where ζ_l is a fixed primitive lth root of unity.

Definition 3.4. A *canonical balanced* $\Gamma_1(l)$*-structure* on $(A/S, i)$ is the following collection of data:

- $\pi : A \to B$ an isogeny of false elliptic curves, of false degree l and kernel K,
- $\pi^t : B \to A$ the dual isogeny, with kernel K',
- $P \in K_1(S)$ a generator of $K_1 = \ker(e : K \to K)$,
- $P' \in K_1'(S)$ a generator of $K_1' = \ker(e^* : K' \to K')$,

such that $\langle P, P' \rangle_\pi = \zeta_l$. Here $\langle P, P' \rangle_\pi$ is the canonical pairing between $\ker\pi$ and $\ker\pi^t$.

Key remark. If S is a $\mathbb{Z}[1/d]$-scheme, and A/S and B/S are false elliptic curves, such that $A[l] \cong B[l]$ as $\mathcal{O}_D$-group schemes, then there is an induced isomorphism between the set of $\Gamma_0(l)$ (resp. $\Gamma_1(l)$, resp. balanced $\Gamma_1(l)$, resp. canonical balanced $\Gamma_1(l)$ if $S/\mathbb{Z}[1/d][\zeta_l]$) structures on A and on B. Moreover, if E/S is a (genuine) elliptic curve, and $A[l] \cong E[l]^2$ as $\mathcal{O}_D$-group schemes, where here $\mathcal{O}_D$ acts via the natural action of $\mathcal{O}_D \otimes \mathbb{F}_l \cong \mathrm{M}_2(\mathbb{F}_l)$ on $E[l]^2$, then we have an induced isomorphism between the set of $\Gamma_0(l)$ (resp. $\Gamma_1(l)$, resp. etc) structures on A, as defined above, and the set of $\Gamma_0(l)$ (resp. $\Gamma_1(l)$, resp. etc) structures on E/S as defined in [KM]. The only structure for which this is not clear given the results of [KM] Chapter 1 is the $\Gamma_0(l)$ structure, because of the cyclicity condition in [KM] for $\Gamma_0(l)$ structures. But in the theory of elliptic curves, all l-isogenies are cyclic by [KM] Corollary 6.8.7.

§4. Analysis of the new structures.

We now wish to prove many results about these structures at l. All the results here are analogues of results proved in [DR]. Again throughout this section, l is a prime not dividing d.

Let S_0 be a base scheme, and let $\mathcal{F}$ be a contravariant functor from false elliptic curves over S_0-schemes to the category of sets. For example, $S_0 = \mathrm{Spec}(\mathbb{Z}[1/d])$ and $\mathcal{F}$ is the functor defined on objects by $\mathcal{F}(A/S) = \{$isomorphism classes of $\Gamma_0(l)$-structures on $A/S\}$, or $S_0 = \mathrm{Spec}(\mathbb{Z}[1/d][\zeta_l])$ and $\mathcal{F}$ is the functor defined by $\mathcal{F} : A/S \mapsto \{$isomorphism classes of canonical balanced $\Gamma_1(l)$-structures on $A/S\}$. We say that $\mathcal{F}$ is *relatively representable* if, given any fixed S/S_0 and any fixed false elliptic curve A/S, the functor from S-schemes to sets sending T/S to $\mathcal{F}(A_T)$ is representable. We now prove some relative representability results.

Proposition 4.1. If $S_0 = \mathrm{Spec}(\mathbb{Z}[1/d])$, and $\mathcal{T} \in \{\Gamma_0(l),\ \Gamma_1(l),\ \text{balanced } \Gamma_1(l)\ \}$, then the functor $\mathcal{F}$ defined by $\mathcal{F}(A/S) = \{$isomorphism classes of $\mathcal{T}$-structures on $A/S\}$ is relatively representable. If $S_0 = \mathrm{Spec}(\mathbb{Z}[1/d][\zeta_l])$, then the functor $\mathcal{F}(A/S) = \{$isomorphism classes of canonical balanced $\Gamma_1(l)$-structures on $A/S\}$ is relatively representable.

Proof. Let $A/S/\mathbb{Z}[1/d]$ be a false elliptic curve. The result for $\mathcal{F} = \Gamma_0(l)$ is a consequence of the representability of a certain Grassmannian. For the map $A[l] \to S$ is finite and locally free, and hence defines a coherent sheaf $\mathcal{A}$ of bi-algebras on S, namely the push-forward of $\mathcal{O}_{A[l]}$. A $\Gamma_0(l)$-structure can be thought of in this context as an $\mathcal{O}_D$-invariant quotient $\mathcal{B}$ of $\mathcal{A}$ which is locally free of rank l^2, and such that the kernel of $\mathcal{A} \to \mathcal{B}$ is a bi-ideal of $\mathcal{A}$. Hence $\mathcal{F}$ is represented by a closed subscheme of the Grassmannian of all rank l^2 quotients of $\mathcal{A}$.

In more concrete terms, the above argument shows the existence of a scheme S'/S together with a $\Gamma_0(l)$-structure $\varpi : A_{S'} \to B$ on $A_{S'}$ which is universal for $\mathcal{F}$, in the sense that every $\Gamma_0(l)$-structure on every extension A_T of A arises as a pullback of ϖ. Hence it is easy to see that the functor from S-schemes to sets sending T/S to the set of pairs consisting of a $\Gamma_0(l)$-structure π, and a homomorphism α from $\mathbb{Z}/l\mathbb{Z}$ to $(\ker(e : \ker\pi \to \ker\pi))(T)$, is representable by $\ker(e : \ker\varpi \to \ker\varpi)$. The relative representability of the $\Gamma_1(l)$-structure follows from this, because there is a closed subscheme of this scheme which is universal for the statement "α is a generator" by [KM] 1.9.1.

The relative representability of the balanced $\Gamma_1(l)$ functor can be deduced in a similar way.

Finally, if S is a scheme over $\mathbb{Z}[1/d][\zeta_l]$, A/S is a false elliptic curve, and S'' represents the functor of balanced $\Gamma_1(l)$ structures on A/S, then there is a closed subscheme of S'' which is universal for the statement "the structure is canonical".

$\square$

Corollary 4.2. If U is compact and open satisfying (i)–(iii) as usual, and l does not divide $M_U d$, then the functor from $\mathbb{Z}[1/M_U d]$-schemes to sets sending S to the set of (isomorphism classes of) false elliptic curves over S equipped with a level U structure and a $\Gamma_0(l)$-structure is representable by a scheme $X^D(U, \Gamma_0(l))$.

Proof. Apply the relative representability result for $\Gamma_0(l)$ to the universal false elliptic curve over $X^D(U)$.

$\square$

Corollary 4.3. If U is as above, and l is a prime not dividing $M_U d$, then the functor from $\mathbb{Z}[1/M_U d][\zeta_l]$-schemes to sets sending S to the set of isomorphism classes of false elliptic curves over S, equipped with a level U structure and a canonical balanced $\Gamma_1(l)$-structure, is representable by a scheme $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))/\mathbb{Z}[1/M_U d][\zeta_l]$.

Proof. Apply the relative representability result for canonical balanced $\Gamma_1(l)$ to the universal false elliptic curve over the pullback $X^D(U)_{\mathbb{Z}[1/M_U d][\zeta_l]}$ of $X^D(U)$.

$\square$

We now show that these new moduli problems are in fact extensions of naïve moduli problems discussed earlier.

Lemma 4.4.
(i) If $S/\mathbb{Z}[1/dl]$ is a scheme, and A/S is a false elliptic curve, then there is a bijection between the $\Gamma_0(l)$-structures on A and the naïve $V_0(l)$-structures on A. Similarly, there is a bijection between the $\Gamma_1(l)$-structures on A and the naïve $V_1(l)$-structures on A.
(ii) If $S/\mathbb{Z}[1/dl][\zeta_l]$ is a scheme, and A/S is a false elliptic curve, then there is a bijection between the canonical balanced $\Gamma_1(l)$-structures on A and the naïve $V_1(l)$-structures on A.

Proof.
(i) Firstly, we note that the functor on S-schemes sending T/S to the set of $\Gamma_0(l)$-structures on A_T is an étale sheaf on S, and the functor on S-schemes sending T/S to the set of naïve $V_0(l)$-structures on A_T is also an étale sheaf. Hence it suffices to establish the required bijection after replacing S by an étale surjective cover of S. So without loss of generality we may assume that $A[l] \cong ((\mathbb{Z}/l\mathbb{Z})^4)_S$. If α is a $V_0(l)$ structure on A/S then we can choose a full level l structure $\beta : (\mathcal{O}_D \otimes \mathbb{Z}/l\mathbb{Z})_S \to A[l]$ which lifts α. The subgroup $\beta(\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix})$ of $A[l]$ is finite and locally free of rank l, and killed by e, and as we showed above, this can be thought of as a $\Gamma_0(l)$ structure on A/S. It is easily checked that this sets up the required bijection between $V_0(l)$-structures and $\Gamma_0(l)$-structures.

A similar argument gives us a bijection between naïve $V_1(l)$-structures and $\Gamma_1(l)$-structures on A/S. Explicitly, the point of exact order l corresponding to the naïve $V_1(l)$-structure β is the point $\beta(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$.

(ii) By (i), it suffices to show that every $\Gamma_1(l)$-structure on A/S can be uniquely extended to a canonical balanced $\Gamma_1(l)$-structure. We see this as follows. Let $\pi : A \to B$ and $P \in \ker(e : \ker \pi \to \ker \pi)$ be a $\Gamma_1(l)$-structure on A. Then the fact that the pairing $\langle , \rangle_\pi : \ker \pi \times \ker \pi^t \to \mu_l$ is perfect and that $P \neq 0$ implies that there exists $P_1 \in \ker(\pi^t)(S)$ such that $\langle P, P_1 \rangle_\pi = \zeta_l \in \mu_l(S)$. Let $P' = (1 - e^*)P_1$. Then $\langle P, P' \rangle = \langle P, (1 - e^*)P_1 \rangle = \langle (1 - e)P, P_1 \rangle = \langle P, P_1 \rangle = \zeta_l$, and we deduce that $\Gamma_1(l)$-structures can be extended to canonical balanced $\Gamma_1(l)$-structures.

Moveover, because l is invertible on S, we see that this extension is unique. For if Q is any other S-valued point of $\ker(e^* : \ker \pi^t \to \ker \pi^t)$ then Zariski locally we have $Q = nP'$ for some integer n, and hence $\langle P, Q \rangle = (\zeta_l)^n$. But $\langle P, Q \rangle = \zeta_l$, and we deduce that n is congruent to 1 mod l, and hence that $Q = P'$.

□

We now prove an analogue of the Serre-Tate theorem, for false elliptic curves. In fact this theorem can be readily deduced from the Serre-Tate theorem for abelian varieties.

Theorem 4.5. Let l be a prime not dividing d. Let R be a ring in which l is nilpotent, let $I \subseteq R$ be a nilpotent ideal, and set $R_0 = R/I$. Let A_0 be a false elliptic curve over $\mathrm{Spec}(R_0)$. Then there is a functorial bijection between the following two sets:

(a) Isomorphism classes of false elliptic curves A/R equipped with an isomorphism $A \times \mathrm{Spec}(R_0) \to A_0$.

(b) Isomorphism classes of l-divisible groups G/R equipped with an action of $\mathcal{O}_D$ (or equivalently an action of $\mathrm{M}_2(\mathbb{Z}_l)$) and an $\mathcal{O}_D$-isomorphism $G \times \mathrm{Spec}(R_0) \to A_0[l^\infty]$.

The bijection is given by the map $A \mapsto A[l^\infty]$.

Proof. It suffices to write down an inverse to this map. Given an l-divisible group as in (b), the Serre-Tate theorem (see for example Theorem 1.2.1 of [Ka]) tells us that there is some abelian surface A/R such that $A \times \mathrm{Spec}(R_0)$ is isomorphic to A_0. It remains to put an action of $\mathcal{O}_D$ on A, lifting the action on A_0. Again we can do this using the Serre-Tate theorem, which implies that if $j \in \mathcal{O}_D$ then the endomorphisms induced by j on G and on A_0 can be glued uniquely to give an endomorphism of A.

□

Corollary 4.6. Let $A/\overline{\mathbb{F}}_l$ be a false elliptic curve, and let $E/\overline{\mathbb{F}}_l$ be an elliptic curve, such that E is ordinary if A is ordinary, and supersingular if A is supersingular. Let W be the Witt vectors of $\overline{\mathbb{F}}_l$, let $\mathbb{A}/W[\![T]\!]$ be the universal formal deformation of A (see [B] §5 Corollaire) and let $\mathbb{E}/W[\![T]\!]$ be the universal formal deformation of E.

(i) $\mathbb{A}[l^\infty]$ is the universal formal deformation of $A[l^\infty]$ (to l-divisible groups with an $\mathcal{O}_D$-action over Artin local W-algebras).

(ii) $\mathbb{A}[l^\infty] \cong \mathbb{E}[l^\infty]^2$ where $\mathcal{O}_D$ acts on $\mathbb{E}[l^\infty]^2$ via the canonical action of $\mathcal{O}_D \otimes \mathbb{Z}_l \cong \mathrm{M}_2(\mathbb{Z}_l)$.

Proof. (i) is immediate from Theorem 4.5. For (ii) it suffices to note that deforming $A[l^\infty]$ with its $\mathcal{O}_D$-action is equivalent to deforming $A[l^\infty]$ with its induced $\mathrm{M}_2(\mathbb{Z}_l)$-action, which is equivalent to deforming $\ker(\tilde{e} : A[l^\infty] \to A[l^\infty])$ where $\tilde{e} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_l)$. Finally, by our choice of E, we have $E[l^\infty]^2 \cong A[l^\infty]$, $E[l^\infty] \cong \ker(\tilde{e} : A[l^\infty] \to A[l^\infty])$, and hence by the Serre-Tate theorem again we deduce that the deformation theories of $A, A[l^\infty]$ (with their $\mathcal{O}_D$-actions), E and $E[l^\infty]$ are all equivalent in this setting.

$\square$

We will use this latter corollary to deduce certain results about Shimura curves from the corresponding results for modular curves. The next theorem is an example of this.

As in [DR] we use the suffix ^{h} to denote the ordinary (at l) locus of a moduli problem. So for example $X^D(U,\Gamma_0(l))^h$ is the scheme obtained by removing the supersingular points of $X^D(U,\Gamma_0(l))$ in the fibre at l.

The following theorem summarises the main results which we shall need about the scheme $X^D(U,\Gamma_0(l))$.

Theorem 4.7.
(i) The scheme $X^D(U,\Gamma_0(l))$ is smooth over $\mathbb{Z}[1/M_U d]$ away from the supersingular points in characteristic l.
(ii) $X^D(U,\Gamma_0(l))/\mathbb{Z}[1/M_U d]$ is proper.
(iii) The map $c : X^D(U,\Gamma_0(l)) \to X^D(U)$ forgetting the $\Gamma_0(l)$ structure is finite.
(iv) c is flat, and $X^D(U,\Gamma_0(l))$ is a regular scheme.
(v) The fibre $X^D(U,\Gamma_0(l))_{\mathbb{F}_l}$ of $X^D(U,\Gamma_0(l))$ is composed of two irreducible components, both isomorphic to $X^D(U)_{\mathbb{F}_l}$, which cross transversally at the supersingular points.

Remark. The proofs of the analogous results in [DR] V.1 go through in many cases. We shall also use the techniques of [KM] Chapter 5.

Proof.
(i) Let $\mathcal{A}(U)$ denote the functor from $\mathbb{Z}[1/M_U d]$-schemes to sets, sending S to the set of false elliptic curves over S equipped with a naïve level U structure and an $\mathcal{O}_D$-invariant subgroup scheme which is locally (which for the duration of this paragraph means "locally for the étale topology") $\mathcal{O}_D$-isomorphic to $(\mu_l)^2$ with its canonical action of $\mathcal{O}_D$ (via the natural action of $\mathrm{M}_2(\mathbb{F}_l)$ and the isomorphism $\mathcal{O}_D \otimes \mathbb{F}_l \cong \mathrm{M}_2(\mathbb{F}_l)$ fixed in section §1). Define $\mathcal{B}(U)$ to be the functor from $\mathbb{Z}[1/M_U d]$-schemes to sets, sending S to the set of false elliptic curves over S equipped with a level U structure and a closed $\mathcal{O}_D$-invariant subgroup scheme locally isomorphic to $(\mathbb{Z}/l\mathbb{Z})^2$. If $(A/S, i)$ is a false elliptic curve with a naïve level U structure α and a closed $\mathcal{O}_D$-invariant subgroup scheme G of $A[l]$ locally isomorphic to $(\mu_l)^2$, then A/G can also be given the structure of a false elliptic curve over S, and the $\mathcal{O}_D$-invariant subgroup scheme $A[l]/G$ is locally isomorphic to $(\mathbb{Z}/l\mathbb{Z})^2$. Moreover, the naïve level U structure on A can be pushed forward to A/G. This induces a natural isomorphism of functors from $\mathcal{A}(U)$ to $\mathcal{B}(U)$. Moreover, by the theory of the Hilbert scheme, $\mathcal{B}(U)$ is representable, and by SGA 3, IX 3.6 bis, $\mathcal{A}(U)$ is formally étale over $X^D(U)$ and hence smooth over $\mathbb{Z}[1/M_U d]$. So both $\mathcal{A}(U)$ and $\mathcal{B}(U)$ are smooth over $\mathbb{Z}[1/M_U d]$. Moreover, $X^D(U,\Gamma_0(l))^h$ is the union of $\mathcal{A}(U)$ and $\mathcal{B}(U)$, and furthermore the intersection of $\mathcal{A}(U)$ and $\mathcal{B}(U)$ is the open subscheme of $X^D(U,\Gamma_0(l))^h$ obtained by removing the fibre at l, which is nothing other than the scheme $X^D(U \cap V_0(l))$ by Lemma 4.4. Hence $X^D(U,\Gamma_0(l))$ is smooth away from the supersingular points.
(ii) This follows from the valuative criterion of properness and the fact that false elliptic curves have potentially good reduction (see section 6 of [B]). One need only check that if $A/\mathcal{O}$ is an abelian scheme over a discrete valuation ring, and π is a $\Gamma_0(l)$ structure on the

generic fibre of A, that π extends uniquely to a $\Gamma_0(l)$ structure on A. This follows from the fact that A is a Néron model of its generic fibre.
(iii) The map c is proper, and so it suffices to check that its geometric fibres are finite. In particular, it suffices to show that if A/k is a false elliptic curve over an algebraically closed field, then there are only finitely many $\Gamma_0(l)$-structures on A/k. We have $A[l] \cong G \times G$, for G of rank l^2, and a $\Gamma_0(l)$ structure on A/k can be thought of as a rank l subgroup of G. There are three cases, which can be checked separately:

(a) The characteristic of k is not equal to l. In this case, G is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^2$ and there are $l+1$ subgroups of G having rank l.

(b) $\mathrm{char}(k) = l$ and A/k ordinary. Then G is isomorphic to $\mathbb{Z}/l\mathbb{Z} \times \mu_l$ and there are only 2 subgroups of G of rank l, namely $\mathbb{Z}/l\mathbb{Z}$ and μ_l.

(c) $\mathrm{char}(k) = l$ and A/k supersingular. Then $G \cong \alpha_{l^2}$ because G is the l-torsion in the unique one-parameter formal Lie group of height 2 over k. Hence there is only one subgroup of G of rank l, namely α_l, the kernel of Frobenius.

Thus c is proper with finite fibres and is hence finite.
(iv) Here, rather than modifying proofs in [DR] and [KM] into this setting, we instead actually *use* results of [KM] to deduce results in our case. In particular, [KM] have to work quite hard to get their Theorem 6.6.1, deducing it from analogous results for $\Gamma_1(l)$ structures. However, we do not have to redo the results of [KM] chapters 5 and 6 in our setting, because we can use [KM] Theorem 6.6.1 itself. Firstly here is a brief summary of what we shall do.

The conditions that we have to check (regularity and flatness) are conditions on the local rings of $X^D(U)$ and $X^D(U, \Gamma_0(l))$. Firstly we reduce the problem to having to check these conditions only on the local rings of closed points with residue field of characteristic l. Then we show that, after strict henselisation and completion (which does not affect the properties we need to check), these latter local rings become isomorphic to the local rings of points on modular curves, where flatness and regularity have already been proved in [DR] or [KM]. Hence we deduce flatness and regularity in our case. Now here are the details.

Consider the subset $\mathcal{Z}$ of $X^D(U)$ consisting of the points $y \in X^D(U)$ such that, for all points x of $X^D(U, \Gamma_0(l))$ lying above y, the local ring $\mathcal{O}_x$ is regular, and flat over $\mathcal{O}_y$. Then $\mathcal{Z}$ is open, as c is a closed map. We must show that $\mathcal{Z} = X^D(U)$. Now because $X^D(U)$ is of finite type over $\mathbb{Z}[1/M_U d]$, we need only prove that $\mathcal{Z}$ contains every closed point of $X^D(U)$. For points of residue characteristic not equal to l this is a consequence of the results of §2, where a $\Gamma_0(l)$ structure can be interpreted as a naïve $V_0(l)$ structure. Hence we are reduced to considering points of residue characteristic l.

Hence we must show that if $y \in X^D(U)$ is a closed point of residue characteristic l, and $x \in X^D(U, \Gamma_0(l))$ lies above it, then the local ring $\mathcal{O}_x$ is regular, and flat over $\mathcal{O}_y$. We shall do this by deducing it from the corresponding result for modular curves, using the ideas on p133 of [KM]. Firstly we reduce the problem to one about complete strictly Henselian local rings. Let $k = \overline{\mathbb{F}}_l$, and let W be the Witt vectors of k. For a local ring A, let $A^{\mathrm{h.s.}}$ denote its strict henselisation, and $\widehat{A}$ its completion.

Let y_0 be any closed point of $X^D(U)$ of residue characteristic l, and let y be any k-valued point of $X^D(U) \times_{\mathbb{Z}} W$ lying over y_0. Then the local rings of y and y_0 are related by $\widehat{\mathcal{O}^{\text{h.s.}}_{X^D(U),y_0}} \cong \widehat{\mathcal{O}_{X^D(U)\times W,y}}$, and it follows from this fact that we are done if we can show that for any k-valued point x of $X^D(U,\Gamma_0(l)) \times W$ lying over y, the complete local ring $\widehat{\mathcal{O}_{X^D(U,\Gamma_0(l))\times W,x}}$ is regular, and flat over $\widehat{\mathcal{O}_{X^D(U)\times W,y}}$.

Because c is finite from (iii), this can be reinterpreted as saying that the scheme

$$(X^D(U,\Gamma_0(l)) \times W) \times_{(X^D(U)\times W)} \operatorname{Spec}(\widehat{\mathcal{O}_{X^D(U)\times W,y}}) \qquad (*)$$

is regular, and flat over $\widehat{\mathcal{O}_{X^D(U)\times W,y}}$. This scheme is nothing other than the scheme relatively representing $\Gamma_0(l)$-structures on the false elliptic curve $\mathbb{A}/\widehat{\mathcal{O}_{X^D(U)\times W,y}}$, the universal formal deformation of the false elliptic curve A/k corresponding to y. Now let E/k be an elliptic curve which is ordinary if A is ordinary, and supersingular if A is supersingular. By Corollary 4.6, $\mathbb{A}[l] \cong \mathbb{E}[l]^2$ where $\mathbb{E}$ is the universal formal deformation of E/k to Artin local W-algebras with residue field k. We deduce that to give a $\Gamma_0(l)$-structure on $\mathbb{A}$ is to give one on $\mathbb{E}$, and hence the scheme $(*)$ above is *isomorphic* to the corresponding scheme denoted $(*)$ on p133 of [KM], for $\mathcal{P} = [\Gamma_0(l)]$ in the notation of [KM]. Now by [KM] Theorem 6.6.1, this scheme is indeed a regular scheme, flat over $\widehat{\mathcal{O}_{X^D(U)\times W,y}}$ (which is isomorphic to $W[\![T]\!]$ by [B] §5 and Proposition 2.5) and hence $\mathcal{Z}$ does indeed contain every closed point with residue characteristic l and we are done.

(v) If S is a scheme over $\mathbb{F}_l$, then the absolute Frobenius map $F_{\text{abs}} : S \to S$ gives us by pullback a diagram

$$\begin{array}{ccc} A^{(l)} & \longrightarrow & A \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_{\text{abs}}} & S \end{array}$$

and hence in the usual way an S-morphism $F : A \to A^{(l)}$. The endomorphisms $i(x)$ of A, for $x \in D$, induce endomorphisms of $A^{(l)}$, and these endomorphisms define the structure of a false elliptic curve on $A^{(l)}$, with respect to which F becomes an isogeny of false elliptic curves, of false degree l. Write V for the dual of F.

There are maps Φ_1 and $\Phi_2 : X^D(U)_{\mathbb{F}_l} \to X^D(U,\Gamma_0(l))_{\mathbb{F}_l}$ defined (functorially) by

$$\Phi_1 : (A,\phi) \mapsto (A,\phi,F)$$

and

$$\Phi_2 : (A,\phi) \mapsto (A^{(l)}, F_*\phi, V),$$

where $F_*\phi$ denotes the push-forward of ϕ defined in §1. Hence we have a map

$$\Phi_1 \amalg \Phi_2 : X^D(U)_{\mathbb{F}_l} \amalg X^D(U)_{\mathbb{F}_l} \to X^D(U,\Gamma_0(l))_{\mathbb{F}_l}.$$

If $c : X^D(U,\Gamma_0(l)) \to X^D(U)$ is the forgetful map, then $c\Phi_1$ is the identity on $X^D(U)_{\mathbb{F}_l}$, and $c\Phi_2$ is the map $(A,\phi) \mapsto (A^{(l)}, F_*\phi)$.

Define $w : X^D(U,\Gamma_0(l)) \to X^D(U,\Gamma_0(l))$ thus: If A/S is a false elliptic curve over a $\mathbb{Z}[1/M_U d]$-scheme, with a level U structure ϕ and an isogeny $\psi : A \to B$ of false degree l, then $w(A,\phi,\psi) = (B, \psi_*\phi, \psi^t)$. Note that $\Phi_2 = w\Phi_1$.

The two maps Φ_1 and Φ_2 are closed immersions, because they have left inverses. Moreover, the only rank l subgroups of $(\mu_l \times \mathbb{Z}/l\mathbb{Z})/\overline{\mathbb{F}}_l$ are μ_l and $\mathbb{Z}/l\mathbb{Z}$, and hence

$$\Phi_1 \amalg \Phi_2 : X^D(U)^h_{\mathbb{F}_l} \amalg X^D(U)^h_{\mathbb{F}_l} \to X^D(U, \Gamma_0(l))^h_{\mathbb{F}_l}$$

is an isomorphism. Hence the images of Φ_1 and Φ_2 are the two irreducible (by Proposition 2.4) components of $X^D(U, \Gamma_0(l))_{\mathbb{F}_l}$. Finally we must show that the intersection of these two components at the supersingular points is transversal. One way of seeing this is as follows: we have a map $c : X^D(U, \Gamma_0(l)) \to X^D(U)$ forgetting the $\Gamma_0(l)$ structure. The map cw is hence also a map from $X^D(U, \Gamma_0(l))$ to $X^D(U)$. Now one sees that the tangent vectors at a supersingular point in characteristic l are distinct, because the one corresponding to Φ_1 is killed by $d(cw)$ and not by $d(c)$, whereas the one corresponding to Φ_2 is killed by is killed by $d(c)$ and not by $d(cw)$.

□

Another method of analysing the supersingular points is by reducing to the classical case, as follows. The spectrum of the completed local ring at any supersingular point x of $X^D(U, \Gamma_0(l))_{\overline{\mathbb{F}}_l}$ is the scheme relatively representing $\Gamma_0(l)$-structures on the universal deformation of $A/\overline{\mathbb{F}}_l$ to Artin local $\overline{\mathbb{F}}_l$-algebras, where $A/\overline{\mathbb{F}}_l$ is the false elliptic curve corresponding to x. But this scheme is isomorphic to the scheme relatively representing $\Gamma_0(l)$-structures on the universal deformation of a supersingular elliptic curve over $\overline{\mathbb{F}}_l$ to Artin local $\overline{\mathbb{F}}_l$-algebras, which is the spectrum of the completed local ring of the classical modular curve classifying elliptic curves with a $\Gamma_0(l)$ structure and some sufficiently strong naïve level structure prime to l. Hence one can again read off certain results on the structure of these complete local rings from classical results.

Before we embark upon an analysis of the structure of $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))$ we have to define one final structure, which only makes sense for false elliptic curves over $\mathbb{F}_l$-schemes.

Recall that if $S/\mathbb{F}_l$ is a scheme and A/S is a false elliptic curve, then the Frobenius morphism $F : A \to A^{(l)}$ is an isogeny of false elliptic curves, of false degree l, and V is defined to be the isogeny dual to F. We have that $\ker(V)$ splits up as $H_1 \times_S H_2$, with $H_1 = \ker(e : \ker V \to \ker V)$ and $H_2 = \ker(1 - e : \ker V \to \ker V)$ as usual.

Definition 4.8. An *Igusa structure* on $(A/S/\mathbb{F}_l, i)$ is a point $P \in H_1(S)$ which is a generator of $H_1 = \ker(e : \ker V \to \ker V)$.

Proposition 4.9. Igusa structures are relatively representable, i.e. if $A/S/\mathbb{F}_l$ is a false elliptic curve, then the functor from S-schemes to sets sending T/S to the set of isomorphism classes of Igusa structures on A_T is representable.

Proof. Indeed, the functor is represented by a closed subscheme of $\ker(e : \ker V \to \ker V)$.

□

As a corollary we see that if $l \nmid M_U$ then there is a scheme $X^D(U, \mathrm{Ig}(l))$ classifying isomorphism classes of false elliptic curves over $\mathbb{F}_l$-schemes equipped with a naïve level U structure and an Igusa structure at l. Analogously to the classical case, the morphism $X^D(U, \mathrm{Ig}(l)) \to X^D(U)_{\mathbb{F}_l}$ is étale over the ordinary locus and totally ramified over the supersingular points. One can see this as follows. If x is an ordinary point of $X^D(U, \mathrm{Ig}(l))_{\overline{\mathbb{F}}_l}$

with residue field $\overline{\mathbb{F}}_l$ and y is its image in $X^D(U)_{\overline{\mathbb{F}}_l}$, then the completed local rings of x and y are isomorphic, by reduction to the classical case. Finally, if $A/\overline{\mathbb{F}}_l$ is a supersingular false elliptic curve, then there is a unique Igusa structure on A, namely the origin. As a corollary, one deduces that $X^D(U, \mathrm{Ig}(l))$ is connected (because supersingular false elliptic curves exist). Moreover, $X^D(U, \mathrm{Ig}(l))$ is regular (again we can see this immediately by reducing to the classical case, because a Noetherian local ring is regular if and only if its completion is regular), and hence irreducible. Lastly, the map $X^D(U, \mathrm{Ig}(l)) \to X^D(U)$ is flat, again because of the analogous result in the classical case.

The second main theorem of this section is the following analogue of [DR] Theorem V.2.12. Recall that we have shown in Corollary 4.3 that if U is a compact open subgroup of $\mathcal{O}_{D,f}^\times$ satisfying the usual conditions (i)–(iii), then the functor sending an $\mathbb{Z}[1/M_U d][\zeta_l]$-scheme S to the set of false elliptic curves $(A/S, i)$ equipped with a naïve level U structure and a canonical balanced $\Gamma_1(l)$-structure is representable by a scheme $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))/\mathbb{Z}[1/M_U d][\zeta_l]$.

Theorem 4.10.
(i) $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))^h \to \mathrm{Spec}(\mathbb{Z}[1/M_U d][\zeta_l])$ is smooth of relative dimension 1.
(ii) $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l)) \to \mathrm{Spec}(\mathbb{Z}[1/M_U d][\zeta_l])$ is proper.
(iii) The natural map $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l)) \to X^D(U, \Gamma_0(l))_{\mathbb{Z}[1/M_U d][\zeta_l]}$ is finite, and hence $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l)) \to X^D(U)_{\mathbb{Z}[1/M_U d][\zeta_l]}$ is also finite.
(iv) The morphism $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l)) \to X^D(U, \Gamma_0(l))_{\mathbb{Z}[1/M_U d][\zeta_l]}$ is flat, and the scheme $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))$ is regular.
(v) The fibre $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))_{\mathbb{F}_l}$ is composed of two irreducible components, both isomorphic to $X^D(U, \mathrm{Ig}(l))$, and which cross transversally at the supersingular points.

Proof. Many of the ideas here are similar to those in the proof of Theorem 4.7.
(i) It suffices to prove that $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))^h \to X^D(U)^h$ is étale. Recall $\mathcal{A}(U)$ and $\mathcal{B}(U)$ defined in the course of the proof of Theorem 4.7. Let $\mathcal{U}(U)$ (resp. $\mathcal{V}(U)$) be the functor on $\mathbb{Z}[1/M_U d][\zeta_l]$-schemes sending a scheme S to the set of isomorphism classes of false elliptic curves A/S equipped with a level U structure and a canonical balanced $\Gamma_1(l)$-structure (π, π^t, P, P') such that $\ker \pi$ is locally (étale) $\mathcal{O}_D$-isomorphic to μ_l^2 (resp. $(\mathbb{Z}/l\mathbb{Z})^2$) with its natural $\mathcal{O}_D$-action (via $\mathrm{M}_2(\mathbb{F}_l)$). Then the natural map $\mathcal{U}(U) \to \mathcal{A}(U)_{\mathbb{Z}[1/M_U d][\zeta_l]}$ (resp. $\mathcal{V}(U) \to \mathcal{B}(U)_{\mathbb{Z}[1/M_U d][\zeta_l]}$) is finite and étale, and $X^D(U, \mathrm{Bal.can}\ \Gamma_1(l))$ is the union of $\mathcal{U}(U)$ and $\mathcal{V}(U)$, glued over $\mathcal{U}(U)[1/l] \cong \mathcal{V}(U)[1/l]$ where $\mathbb{Z}/l\mathbb{Z}$ and μ_l become canonically isomorphic via the map $1 \mapsto \zeta_l$.
(ii) Again this follows from the valuative criterion of properness, once one checks that canonical balanced $\Gamma_1(l)$ structures extend uniquely from the generic fibre to the whole of a discrete valuation ring, after base extension if necessary. More precisely, the theorem of potentially good reduction in section 6 of [B] reduces us to showing the following: if K is a field and a $\mathbb{Z}[1/d][\zeta_l]$-algebra, and K is equipped with a discrete valuation, $\mathcal{O}_K$ is the integers of K, and $A/\mathcal{O}_K$ is a false elliptic curve, then a canonical balanced $\Gamma_1(l)$-structure on A_K extends uniquely to a canonical balanced $\Gamma_1(l)$ structure on A. That the structure extends uniquely to a balanced $\Gamma_1(l)$-structure is immediate from the Néron property of $A/\mathcal{O}_K$. Now one observes that the pairings defined in [O] are compatible with base extension, and hence that the extended structure is still canonical.

(iii) It is an easy check that $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l)) \to X^D(U, \Gamma_0(l))_{\mathbb{Z}[1/M_U d][\zeta_l]}$ is proper with finite fibres and is hence finite.
(iv) As in Theorem 4.7 we can deduce regularity and flatness from the corresponding results in the modular curve case.
(v) The generic fibre of $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))$ is isomorphic to $X^D(U \cap V_1(l))_{\mathbb{Q}} \times \mathbb{Q}(\zeta_l)$, where $X^D(U \cap V_1(l))_{\mathbb{Q}}$ is the canonical model defined by Shimura. We can identify the special fibre at l of $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))$ with the union of two copies of $X^D(U, \mathrm{Ig}(l))_{\mathbb{F}_l}$ by giving two morphisms from $X^D(U, \mathrm{Ig}(l))_{\mathbb{F}_l}$ to $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))_{\mathbb{F}_l}$ in the following way. The morphisms are induced by the two morphisms of moduli problems

$$j_1 : (A/S, \phi, P \in A^{(l)}(S)) \mapsto (A/S, \phi, F, V, 0, \tau(P))$$

and

$$j_2 : (A/S, \phi, P \in A^{(l)}(S)) \mapsto (A^{(l)}/S, F_*\phi, V, F, P, 0),$$

where τ is an appropriate matrix, for example $\tau = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{t} \right)^{-1} \in \mathrm{GL}_2(\mathbb{F}_l)$, where $\bar{t}$ is the image of the t of §1 in $\mathrm{M}_2(\mathbb{F}_l)$. Note that

$$j_1 \amalg j_2 : X^D(U, \mathrm{Ig}(l)) \amalg X^D(U, \mathrm{Ig}(l)) \to X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))_{\mathbb{F}_l}$$

is surjective on $\overline{\mathbb{F}}_l$-valued points, by checking the two cases. If $A/\overline{\mathbb{F}}_l$ is an ordinary false elliptic curve, then the finite group scheme $A[l]$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z} \times \mu_l)^2$ and hence at least one of P and P' must be the zero section. If $A/\overline{\mathbb{F}}_l$ is supersingular, then $F = V$ and $P = P' = 0$ is the only canonical balanced $\Gamma_1(l)$-structure on A and we see that this structure is in the image of both j_1 and j_2. Moreover, j_1 and j_2 are certainly monomorphisms by their definitions, and hence closed immersions because they are proper, being $X^D(U)$-maps between finite flat $X^D(U)_{\mathbb{F}_l}$-schemes (after choosing appropriate maps down to $X^D(U)$). We deduce that the special fibre at l of $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))$ has two irreducible components, which intersect at the supersingular points. One can again check that the intersection at the supersingular points is transversal by using the analogous result for the modular curve case.

□

Proofs of analogous results for totally real fields other than $\mathbb{Q}$ have been given by Carayol in [C] and by Jarvis in [J].

If we set $U = V_1(N)$ for some $N \geq 4$ then $X^D(U, \mathrm{Bal.can}\,\Gamma_1(l))_{\mathbb{Z}_l[\zeta_l]}$ is an analogue of the well-known regular model of $X_1(Nl)$ over $\mathbb{Z}_l[\zeta_l]$. We see from the above theorem and [DR] that these two curves have very similar properties in characteristic l. There is one aspect of the theory of these Shimura curves where the analogy with modular curves is less than perfect. There is no need for a theory of compactification in this setting, because the Shimura curves over $\mathbb{C}$ arising from indefinite non-split quaternion algebras are already compact Riemann surfaces. Hence there are no cusps in this setting. Whilst saving us the details of compactification, it also means that there is no obvious notion of a q-expansion.

§5. The Hasse invariant.

As an example of how the analogy with modular curves continues, we shall construct a section $a \in H^0(X^D(V_1(N), \mathrm{Ig}(l))_{\mathbb{F}_l}, \omega)$ which plays the role of the section a defined in §5 of [G], and in [KM].

Define $X_1^D(N)_{\mathbb{Z}_l} := X^D(V_1(N))_{\mathbb{Z}_l}$ and $I_1^D(N)_{\mathbb{F}_l} := X^D(V_1(N), \mathrm{Ig}(l))_{\mathbb{F}_l}$. Recall from §4 that there is a natural map $\pi : I_1^D(N)_{\mathbb{F}_l} \to X_1^D(N)_{\mathbb{F}_l}$ forgetting the Igusa structure, which is étale away from the supersingular points, and totally ramified of degree $l-1$ at the supersingular points. In [DT] §4 an invertible sheaf $\omega = \omega_{X_1^D(N)_{\mathbb{Z}_l}}$ is defined on $X_1^D(N)_{\mathbb{Z}_l}$, analogous to the sheaf $\omega_{X_1(N)_{\mathbb{Z}_l}}$ on $X_1(N)_{\mathbb{Z}_l}$ whose sections give us modular forms. If $pr : A_{\mathrm{univ}} \to X_1^D(N)_{\mathbb{Z}_l}$ is the universal false elliptic curve over $X_1^D(N)_{\mathbb{Z}_l}$, then $\omega_{X_1^D(N)_{\mathbb{Z}_l}} = \tilde{e} pr_* \Omega^1_{A_{\mathrm{univ}}/X_1^D(N)_{\mathbb{Z}_l}}$, where $\tilde{e} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_l) \cong \mathcal{O}_D \otimes \mathbb{Z}_l$. Define $\omega_{X_1^D(N)_{\mathbb{F}_l}}$ to be the pullback of this sheaf to the special fibre, and define $\omega_{I_1^D(N)}$ to be $\pi^* \omega_{X_1^D(N)_{\mathbb{F}_l}}$. Let ss denote the divisor of supersingular points on $I_1^D(N)$, and let s be the degree of this divisor, so s is the number of points in $I_1^D(N)(\overline{\mathbb{F}}_l)$ (or equivalently in $X_1^D(N)(\overline{\mathbb{F}}_l)$) corresponding to supersingular false elliptic curves over $\overline{\mathbb{F}}_l$.

Proposition 5.1. The invertible sheaf $\omega_{I_1^D(N)}$ on $I_1^D(N)$ has degree s. We have

$$\Omega^1_{I_1^D(N)} \cong \omega^{\otimes 2}_{I_1^D(N)}((l-2)\mathrm{ss}).$$

Proof. From Lemma 6 of [DT], we have $2s = (l-1) \deg \Omega_{X_1^D(N)/\mathbb{F}_l}$, and from Lemma 7 of [DT] we deduce $\omega^{\otimes 2}_{X_1^D(N)} \cong \Omega_{X_1^D(N)/\mathbb{F}_l}$ (We remark that in [DT] l is assumed to be odd, and $\tilde{e}$ is chosen in $\mathrm{M}_2(\mathbb{Z}_l)$ such that $\tilde{e}^* = \tilde{e}$. This may not be possible if $l = 2$, but in any case it is sufficient for the proof of [DT] Lemma 7 that $\tilde{e}^*$ be conjugate to $\tilde{e}$, which is always true regardless of the value of l). Hence

$$\begin{aligned} 2s &= (l-1) \deg \Omega_{X_1^D(N)} \\ &= (l-1)(2 \deg \omega_{X_1^D(N)}) \end{aligned}$$

and so $\deg \omega_{I_1^D(N)} = (l-1) \deg \omega_{X_1^D(N)} = s$.

Now by the Riemann-Hurwitz formula, we have

$$\begin{aligned} \Omega_{I_1^D(N)} &\cong \pi^* \Omega_{X_1^D(N)}((l-2)\mathrm{ss}) \\ &\cong \pi^*(\omega^{\otimes 2}_{X_1^D(N)})((l-2)\mathrm{ss}) \\ &\cong \omega^{\otimes 2}_{I_1^D(N)}((l-2)\mathrm{ss}). \end{aligned}$$

□

The following is the analogue of the section a defined in [G] §5.

Theorem 5.2. There is a non-zero global section a of $\omega_{I_1^D(N)}$ with simple zeros at precisely the supersingular points.

Proof. We define a analogously to [KM] Section 12.8. Firstly we construct, given a false elliptic curve $(A/S, i)$ over $S/\mathbb{F}_l$ with an Igusa structure $P \in \ker(V|_{A^{(l)}})(S)$, an element of $H^0(A, \Omega_{A/S})$, thus. The point P gives rise, by duality, to a map

$$\phi_P : \ker(F|_A) \to \mathbb{G}_m.$$

The invariant differential dX/X on $\mathbb{G}_m$ gives rise by pullback to an invariant differential $\phi_P^*(dX/X)$ on $\ker(F)$, which is the reduction of a unique invariant differential ν on A. This construction gives us, for any (A, i) over $S/\mathbb{F}_l$ with an Igusa structure P, an element of $H^0(A, \Omega_{A/S}) = H^0(S, pr_*\Omega_{A/S})$. Applying $\tilde{e}$ to this section gives us a section of $H^0(S, \tilde{e}pr_*\Omega_{A/S})$. If $S = I_1^D(N)$ with its universal false elliptic curve (A, i, ϕ, P), then this construction defines the global section a which we seek. Note that if $A/\overline{\mathbb{F}}_l$ is supersingular then $P = 0$ and hence the map ϕ_P is 0, so a vanishes at every supersingular point. On the other hand, if $A/\overline{\mathbb{F}}_l$ is ordinary, then the construction gives us a non-zero element of $H^0(S, \tilde{e}pr_*\Omega_{A/S})$ and hence a is not the zero section of $\omega_{I_1^D(N)}$.

To show that a has a simple zero at each supersingular point, we use a counting argument. We have shown that a has zeros at every supersingular point, and by Proposition 5.1 the degree of $\omega_{I_1^D(N)}$ is equal to s, the number of supersingular points. Hence a must have a simple zero at each supersingular point.

$\square$

In fact, many of the geometrical constructions and results of [G] and [E] may be generalised to this setting. See [Bu] where more examples are given.

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