Classification of Stopping Times

Recall that we 'do' our stochastic calculus 'on' a <u>stochastic base</u>: $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t), [0,T])$, where $(\Omega, \mathcal{F}_T, \mathbb{P})$ is a (complete) Probability Space, $(\mathcal{F}_t)$, $t \in [0,T]$ is a(n increasing) filtration of σ-fields in $\mathcal{F}_T$ such that:

(i) For $0 \leq s \leq t \leq T$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

(ii) $\bigcup_{t \in [0,T)} \mathcal{F}_t$ generates the σ-field $\mathcal{F}_T$

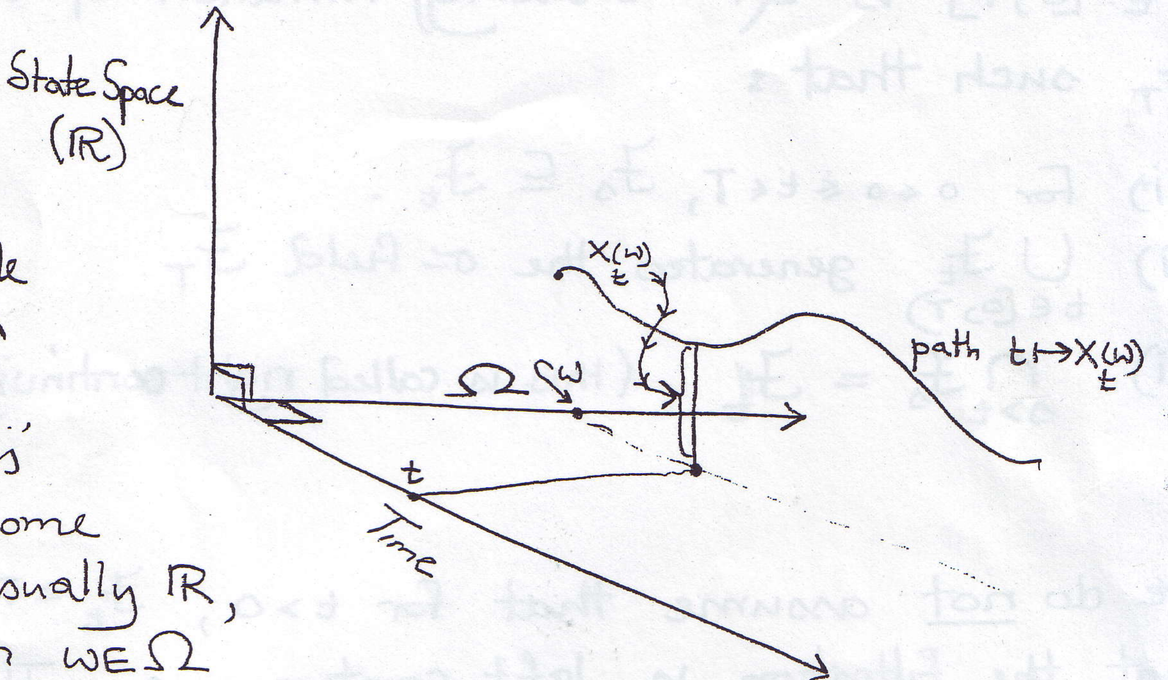
(iii) $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$ (this is called right continuity of the filtrati

We do <u>not</u> assume that for $t>0$, $\mathcal{F}_t = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right)$, i.e.) that the filtration is <u>left continuous</u>. This is an important feature, on some left continuous filtration the ideas presented below, which distinguish between various types of stopping times, all marry into one case and certain kinds of stopping times don't exist. The structure above is general enough to include processes such as Brownian Motion and the Poisson Process living on the same stochastic base. It might be used, for example, when one has a financial asset whose value changes continuously except at certain (random) times where it has discontinuous changes.

A stopping time, τ, is a map from Ω into $[0,T]$, the time set. In order for $\tau : \Omega \to [0,T]$ to be a stopping time we must have, for $t \in [0,T]$,

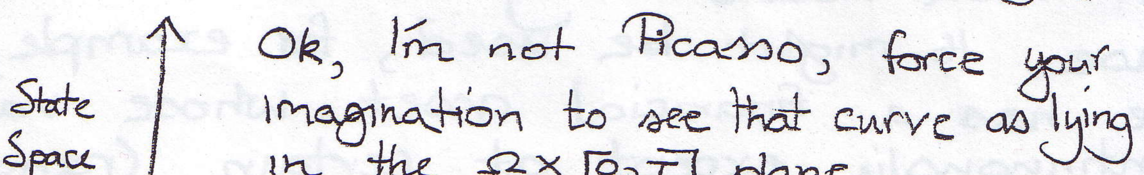
$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t .$$

hen $(\mathcal{F}_t)$ is right continuous this condition is equival-
nt to $\{\omega : \tau(\omega) < t\} \in \mathcal{F}_t$, for $t \in [0,T]$.
At this point it is worth trying to 'expand' our ideas of random times and processes: this is nothing more than taking the definitions seriously.



The idea is simple and obvious. A process lives on $[0,T] \times \Omega$. It's values lie in some state space, usually $\mathbb{R}$, and from each $\omega \in \Omega$ there 'emanates' a path, $t \mapsto X_t(\omega)$.

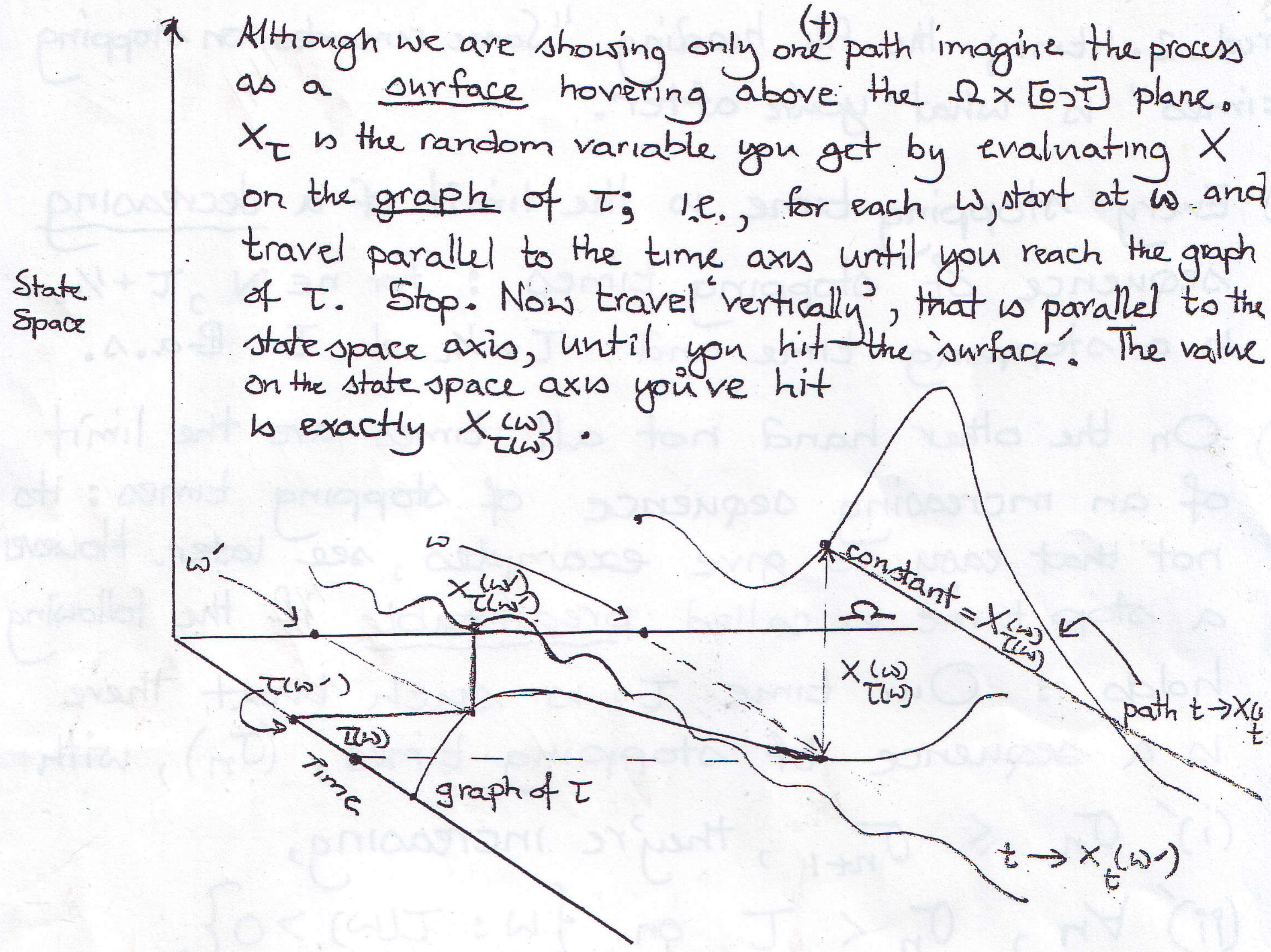
All I want to suggest is that when you think about processes you adopt something like the situation portrayed in the diagram above. So what has this to do with stopping times? Well first of all let τ be a stopping time, it has a graph



Ok, I'm not Picasso, force your imagination to see that curve as lying in the $\Omega \times [0,T]$ plane.

If we combine together the two previous diagrams we can represent how stopping times and processes interact.

Although we are showing only one(†) path imagine the process as a surface hovering above the $\Omega \times [0,T]$ plane. X_τ is the random variable you get by evaluating X on the graph of τ; i.e., for each ω, start at ω and travel parallel to the time axis until you reach the graph of τ. Stop. Now travel 'vertically', that is parallel to the state space axis, until you hit the 'surface'. The value on the state space axis you've hit is exactly $X^{(\omega)}_{\tau(\omega)}$.



The process stopped at time τ, X_τ, is a random variable. A related idea is the stopped process (no wonder this subject is confusing – 'process stopped', 'stopped process', ...): formally the stopped process, X^τ, is defined as $X^{(\omega)}_{\tau(\omega)\wedge t}$, for $\omega \in \Omega$. Looking at the diagram, right hand path, the path of X^τ emanating from ω is just the path of X up to $\tau(\omega)$ but for $t > \tau(\omega)$ the path is "frozen" at the value $X^{(\omega)}_{\tau(\omega)}$.

One last point before we move on. The stochastic interval, $[0,\tau]$, is the area of the $[0,T] \times \Omega$ plane trapped between the Ω-axis and the graph of τ. Open and half-open intervals are defined in the obvious way.

(†) two, I added another one.

(iii) Some stopping times are not predictable but they are, in a fashion, patched together from bits of predictable stopping times. This idea isn't altogether correct: more <u>precisely</u>, imagine in the $[0,T]\times\Omega$ plane a sequence of graphs of predictable times, σ_n, say. Suppose τ is a time whose graph is contained in the <u>union</u> of the graphs of the σ_n's. Then we call τ an <u>accessible</u> stopping time. In fact this idea is usually <u>weakened</u> to allow a bit of the graph of τ to lie outside of the union of the graphs of the σ_n's, but this bit has to be inside an <u>evanescent set</u>. An evanescent set in $[0,T]\times\Omega$ is a special kind of set with zero (product) measure. A process is evanescent if $\mathbb{P}$ a.s. every path is the zero function. A set is evanescent if the indicator function (on $[0,T]\times\Omega$) of the set is an evanescent process.

(iv) Any stopping time which takes at most a countable number of values is accessible

(v) τ is accessible if there is a sequence, (σ_n), comprising predictable times, $\sigma_1, \sigma_2, \ldots$, such that

$$\mathbb{P}\Big(\bigcup_{k=1}^{\infty}\{\omega : \sigma_k(\omega) = \tau(\omega) < \infty\}\Big) = \mathbb{P}\{\omega : \tau(\omega) < \infty\}.$$

So, as $\{\omega : \sigma_k(\omega) = \tau(\omega) < \infty\} \subseteq \{\omega : \tau(\omega) < \infty\}$ then $\bigcup_{k=1}^{\infty}\{\omega : \sigma_k(\omega) = \tau(\omega) < \infty\}$ is 'more or less' $\{\omega : \tau(\omega) < \infty\}$.

(vi) The time of the first jump of a Poisson Process is not an accessible stopping time. A stopping time, τ, is totally inaccessible if for every predictable time, σ,

$$\mathbb{P}\{\omega : \tau(\omega) = \sigma(\omega) < \infty\} = 0.$$

As you can see, if $\mathbb{P}\{\tau < \infty\} > 0$ then no sequence of predictable times has a hope of satisfying the condition in item (v) above because the union of (a countable number of) nulls sets is null.

(vii) There is an intermediate possibility for a stopping time: Part of its graph might lie in the union of the graphs of a sequence of predictable stopping time while the rest of its graph doesn't. When I say 'part' here I'm covering up some technical details, but perhaps the following makes it clear.

First, if τ is a time and $A \in \mathcal{F}_\tau$ then τ_A is defined as

$$\tau_A(\omega) = \begin{cases} \tau & \text{if } \omega \in A \\ \infty & \text{if } \omega \in \Omega \setminus A. \end{cases}$$

Theorem Let τ be a stopping time. There are disjoint sets, $A, B \in \mathcal{F}_\tau$ with $A \cup B = \{\tau < \infty\}$ - $\mathbb{P}$ a.s. And τ_A is totally inaccessible, τ_B is accessible and

$$\tau(\omega) = \min\{\tau_A(\omega), \tau_B(\omega)\} \text{ - } \mathbb{P} \text{ a.s.}$$

Proof: In the books.