Let T_0 and T_1 be successive time instants, $T_0 < T_1$, and let $B(T_0, T_1)$ be the price at time T_0 of a zero coupon bond maturing at time T_1. So this bond has value 1 at time T_1. The existence of this bond implies an interest rate for the period T_0 to T_1. A unit of cash invested in the bond at time T_0 (buys $\frac{1}{B(T_0,T_1)}$ bonds and so) matures to yield $\frac{1}{B(T_0,T_1)}$ in cash at time T_1. We can write $L(T_0)$ for the 'interest rate' implied by the bond so that

$$\frac{1}{B(T_0,T_1)} = 1 + \overset{\theta_{T_0 T_1}}{L(T_0)(T_1 - T_0)} . \qquad (1)$$

Aside: The return on our unit of cash invested at T_0 into the bond is, $\left(\frac{1}{B(T_0,T_1)} - 1\right)$. If $L(T_0)$ is the interest rate per unit of time and $T_1 - T_0$ is expressed in this unit then $\frac{\frac{1}{B(T_0,T_1)} - 1}{1} = L(T_0)(T_1 - T_0)$.

When an interest rate swap is undertaken a finite set of dates is agreed, $T_0, T_1, T_2, \ldots, T_n$, (sometimes called the tenor of the arrangement) these lie 'in the future'. Over the time period T_{j-1} to T_j the floating interest rate is obtained from the price, $B(T_{j-1}, T_j)$, of a zero coupon bond maturing at time T_j. So the time T_{j-1} price, $B(T_{j-1}, T_j)$, gives us $L(T_{j-1})$ and

$$\frac{1}{B(T_{j-1},T_j)} = 1 + \overset{\theta_{T_{j-1} T_j}}{(T_j - T_{j-1})} L(T_{j-1}) . \qquad (2)$$

In addition to this a prearranged fixed interest rate, k, say obtains over each of the time periods, $[T_{j-1}, T_j]$, $j = 1, \ldots, n$. The swap works like this. There are two parties who agree a notional principal amount, N. At times $T_1, T_2, \ldots$, typically T_j, $j = 1, 2, \ldots n$, one party receives $NL(T_{j-1})(T_j - T_{j-1})^{\theta_{T_{j-1}T_j}}$ and pays $k(T_j - T_{j-1})N$. Obviously we can set $N = 1$, for simplicity. 40 pp 39 From the point of view of the party receiving $L(T_{j-1})(T_j - T_{j-1})$ and paying $k(T_j - T_{j-1})$ we can calculate the value at time t, $FS_t(k)$, of this arrangement. At time T_j the net funds received are

$$\left(L(T_j) - k\right)\overset{\theta_{T_{j-1},T_j}}{\left(T_j - T_{j-1}\right)} \qquad (3)$$

we invest this sum in the riskless bond $B(t)$ (cash account) (borrowing if it is negative) and wait until time T_n, the end of the swap arrangement. So this generates

$$\frac{\left(L(T_j)-k\right)\overset{\theta_{T_{j-1},T_j}}{\left(T_j - T_{j-1}\right)}}{B(T_j)} \qquad (4)$$

number of bonds which have value

$$\frac{\left(L(T_j)-k\right)\overset{\theta_{T_{j-1},T_j}}{\left(T_j - T_{j-1}\right)}}{B(T_j)} B(T_n) \qquad (5)$$

at time T_n. Adding up over j gives

$$\sum_{j=1}^{n} \frac{\left(L(T_j)-k\right)\overset{\theta_{T_{j-1},T_j}}{\left(T_j - T_{j-1}\right)}}{B(T_j)} B(T_n) \qquad (6)$$

as the 'payoff' at time T_n of this arrangement. If we are working in a perfect market then there is a probability measure $\mathbb{P}$ such that the discounted value of this claim is a $\mathbb{P}$ martingale, this means that

$$\frac{FS_t(k)}{B_t} = M_t^{\mathbb{P}}\left(\frac{1}{B_{T_n}} \sum_{j=1}^{n} \frac{\left(L(T_j)-k\right)\left(T_j - T_{j-1}\right) B(T_n)}{B(T_j)}\right)$$

that is,

$$FS_t(k) = M_t^{\mathbb{P}}\left(\sum_{j=1}^{n} \frac{\left(L(T_j)-k\right)\left(T_j - T_{j-1}\right)}{[illegible]} \frac{B(t)}{B(T_j)}\right)$$

call these last two equations (7) and (8) respectively.

The kind of swap arrangement we have described here is called

The [illegible]
called a "forward start payer swap settled in arrears". It is Forward start because the time t considered is taken to be before the initiation of the contract. It's called a payer swap, by convention, because our party is paying the fixed component with fixed interest rate, k. It is settled in arrears because payments occur at the end of the predetermined time periods $[T_{j-1}, T_j]$.

Some terminology: The number of payments, n, is often called the *length* (or tenor) of the swap, motivated probably by the practise of making the length of each interval, $[T_{j-1}, T_j]$, the same. The interval $[T_{j-1}, T_j]$ is the "j-th accrual period". Dates, $T_1, T_2, \ldots, T_n$, are the *settlement dates* while $T_0, \ldots, T_{n-1}$ are called *reset dates*, because the floating rate may change at these times. Date T_0 is called the start of the swap.

It can be that the arrangements of the swap are slightly different. We look at this a little later, first though:

Let us suppose (as is often the case) that the time intervals $[T_{j-1}, T_j]$ have constant length, then $\delta = T_j - T_{j-1}$, $1 \leq j \leq n$. Then, recalling that

$$\frac{1}{B(T_{j-1}, T_j)} = 1 + \delta L(T_{j-1})$$

We get

$$FS_t(k) = \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(\left(\left(\frac{1}{B(T_{j-1}, T_j)} - 1\right) - k\delta\right)\frac{B(t)}{B(T_j)}\right)$$

$$= \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(\left(\frac{1}{B(T_{j-1}, T_j)} - \bar{\delta}\right)\frac{B(t)}{B(T_j)}\right) \qquad (10)$$

where $\bar{\delta} = 1+k\delta$. So

$$FS_t(k) = \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(B(T_{j-1},T_j)^{-1} \frac{B_t}{B_{T_{j-1}}} M_{T_{j-1}}^{\mathbb{P}}\left(\frac{B_{T_{j-1}}}{B_{T_j}}\right)\right) - \bar{\delta} M_t^{\mathbb{P}}\left(\frac{B_t}{B_{T_j}}\right) \quad , \quad (11)$$

since $t < T_{j-1}$ + $M_t \leq M_{T_{j-1}}$. Now under $\mathbb{P}$ the discounted <u>value of claims</u> are martingales, therefore

$$\frac{B(t,T_{j-1})}{B_t} = M_t^{\mathbb{P}}\left(\frac{B(T_{j-1},T_{j-1})}{B_{T_{j-1}}}\right) = M_t\left(\frac{1}{B_{T_{j-1}}}\right) \quad (12)$$

so

$$B(t,T_{j-1}) = M_t^{\mathbb{P}}\left(\frac{B_t}{B_{T_{j-1}}}\right) \quad (13)$$

and for "$j-1 = j$" $\quad B(t,T_j) = M_t^{\mathbb{P}}\left(\frac{B_t}{B_{T_j}}\right)$ so

$$M_{T_{j-1}}^{\mathbb{P}}\left(\frac{B_{T_{j-1}}}{B_{T_j}}\right) = B(T_{j-1},T_j) .$$

Hence

$$FS_t(k) = \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(\frac{B_t}{B_{T_{j-1}}}\right) - \bar{\delta} M_t^{\mathbb{P}}\left(\frac{B_t}{B_{T_j}}\right)$$

$$= \sum_{j=1}^{n} B(t,T_{j-1}) - \bar{\delta} B(t,T_j) \quad (14)$$

$$= \left(B(t,T_0) - \bar{\delta} B(t,T_1)\right) + \left(B(t,T_1) - \bar{\delta} B(t,T_2)\right) + \cdots$$

$$\cdots + \left(B(t,T_{n-1}) - \bar{\delta} B(t,T_n)\right)$$

recall $\bar{\delta} = 1+k\delta$,

$$= B(t,T_0) - k\delta B(t,T_1) - k\delta B(t,T_2) - \cdots - k\delta B(t,T_n)$$

$$= B(t,T_0) - \sum_{j=1}^{n-1} (k\delta) B(t,T_j) - \bar{\delta} B(t,T_n) .$$

This exhibits our 'forward swap settled in arrears' as a contract where one receives a zero coupon bond and has to deliver a coupon bond, payments being $k\delta$ for $n-1$ times and $1+k\delta$ at "expiry".

As we remarked swaps can have arrangements which differ from those outlined above. Some swaps may be settled in advance. What this means is that the reset dates are also settlement dates. So at times $T_0, T_1, \ldots, T_{n-1}$, a cash flow occurs as well as these times determining the 'implied interest rate' for the forthcoming period. The cash flow that occurs must be consistent with the swap settled in arrears if they are both available simultaneously otherwise an arbitrage is possible: so in this case the floating payment over $[T_{j-1}, T_j]$ will be $\dfrac{L(T_{j-1})\delta}{(1+L(T_{j-1})\delta)}$ while the 'fixed' payment will be $k\delta(1+L(T_{j-1})\delta)^{-1}$. Each of these payments amount to payments of $L(T_{j-1})\delta$ and $k\delta$ relative to T_j (imagine these cash flows left in the bond maturing at time T_j). So the swap settled in advance should have exactly the same value as that settled in arrears, indeed: at time T_{j-1} the net funds received are,

$$\frac{(L(T_{j-1})-k)\delta}{(1+L(T_{j-1})\delta)}.$$

We invest in the riskless bond until time T_n, so this generates

$$\frac{(L(T_{j-1})-k)\delta}{(1 + L(T_{j-1}) \delta)B(T_{j-1})}$$

number of bonds which have value

$$\frac{(L(T_{j-1})-k)\delta}{(1+L(T_{j-1})\delta)B(T_{j-1})} B(T_n)$$

at time T_n. Adding up over $0 \leq j \leq n-1$ gives

$$\sum_{j=1}^{n} \frac{(L(T_{j-1})-k)\delta\, B(T_n)}{(1+L(T_{j-1})\delta)\, B(T_{j-1})}$$

as the 'payoff' at time T_n. Writing $FS_t^*(k)$ as the value at time t of this arrangement then under risk-neutral probability, $\mathbb{P}$,

$$\frac{FS_t^*(k)}{B_t} = M_t^{\mathbb{P}}\left(\frac{1}{B(T_n)} \sum_{j=1}^{n} \frac{(L(T_{j-1})-k)\delta\, B(T_n)}{(1+L(T_{j-1})\delta)B(T_{j-1})} \right) \quad (15)$$

$$FS_t^*(k) = M_t^{\mathbb{P}}\left(\sum_{j=1}^{n} \frac{(L(T_{j-1})-k)\delta\, B(t)}{(1+L(T_{j-1})\delta)\, B(T_{j-1})} \right) \quad (16)$$

but $1+L(T_{j-1})\delta = B(T_{j-1}, T_j)^{-1}$ so

$$= M_t^{\mathbb{P}}\left(\sum_{j=1}^{n} \frac{B(t)(L(T_{j-1})-k)\delta\, B(T_{j-1}, T_j)}{B(T_{j-1})} \right)$$

$$= \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(\frac{B(t)(L(T_{j-1})-k)\delta}{B(T_{j-1})} M_{T_{j-1}}^{\mathbb{P}}\left(\frac{B(T_{j-1})}{B(T_j)} \right) \right)$$

$$= \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(\frac{B(t)}{B(T_j)} (L(T_{j-1})-k)\delta \right)$$

$$= FS_t(k) . \quad (16)$$

t

In this last example the floating rate determined by the zero coupon bonds was used to discount the 'arrears cash flow' to yeild the 'advanced cash flow'. One could use the fixed rate k to discount the fixed payment and the floating rate to discount the floating payment. This results in cash flows of $L(T_{j-1})\delta(1+L(T_{j-1})\delta)^{-1}$ and $k\delta(1+k\delta)^{-1}$. The values of such an arrangement at time t, after a little rearrangement, is

$$FS_t^{**}(k) = \sum_{j=1}^{n} M_t^{\mathbb{P}}\left(\frac{B_t}{B_{T_{j-1}}}\left(\frac{L(T_{j-1})\delta}{(1+L(T_{j-1})\delta)} - \frac{k\delta}{(1+k\delta)}\right)\right) \quad (17)$$

$$= \sum_{j=1}^{n} \frac{1}{(1+k\delta)} M_t^{\mathbb{P}}\left(\left(\frac{B_t}{B_{T_{j-1}}}\right)\left(\frac{L(T_{j-1})\delta(1+k\delta)}{(1+L(T_{j-1})\delta)} - k\delta\right)\right)$$

$$= \sum_{j=1}^{n} \frac{1}{(1+\delta k)} M_t^{\mathbb{P}}\left(\left(\frac{B_t}{B_{T_{j-1}}}\right)\left(\left(\frac{1}{B(T_{j-1},T_j)} - 1\right)B(T_{j-1},T_j)(1+k\delta) - k\delta\right)\right)$$

$$= \sum_{j=1}^{n} \frac{1}{\tilde{\delta}} M_t^{\mathbb{P}}\left(\left(\frac{B_t}{B_{T_{j-1}}}\right)\left((1-B(T_{j-1},T_j))(1+k\delta) - k\delta\right)\right) \quad (18), \quad \tilde{\delta} = 1+k\delta,$$

$$= \sum_{j=1}^{n} \frac{1}{\tilde{\delta}} M_t^{\mathbb{P}}\left(\left(\frac{B_t}{B_{T_{j-1}}}\right)\left(1-B(T_{j-1},T_j) - B(T_{j-1},T_j)k\delta + k\delta - k\delta\right)\right)$$

$$= \sum_{j=1}^{n} \frac{1}{\tilde{\delta}} M_t^{\mathbb{P}}\left(\left(\frac{B_t}{B_{T_{j-1}}}\right)\left(1 - B(T_{j-1},T_j)\tilde{\delta}\right)\right)$$

$$= \sum_{j=1}^{n} \frac{1}{\tilde{\delta}} \left\{M_t^{\mathbb{P}}\left(M_{T_{j-1}}^{\mathbb{P}}\left(B_t/B_{T_{j-1}}\right)\right) - M_t^{\mathbb{P}}\left(B(T_{j-1},T_j)\tilde{\delta}\right)\right\}$$

$$= \sum_{j=1}^{n} \frac{1}{\tilde{\delta}} M_t^{\mathbb{P}}\left(B(t,T_{j-1}) - \tilde{\delta}B(T_{j-1},T_j)\right)$$

$$= \frac{1}{\tilde{\delta}} \sum^{n} M_t^{\mathbb{P}}\left(B(t,T_{j-1}) - \tilde{\delta}B(T_{j-1},T_j)\right) \quad (19)$$, compare with (14).

So this amounts to the forward start payer swap settled in arrears and discounted for a single time period at the rate k.

There is a feature of swap we must discuss. They are set up so that their value at initiation is zero. We will consider only interest rate swaps settled in arrears. The way in which the swap is set up to have zero value is by choice of k, the fixed interest rate. So we define the forward swap rate, $k(t,T,n)$, at time t for the (future) date T to be that value of the fixed rate, k, for which $FS_t(k) = 0$. From the last equality of equations (14) we get (here $T_0 = T$)

$$0 = B(t,T) - \sum_{j=1}^{n-1} (k\delta)B(t,T_j) - \tilde{\delta}(B_t, T_n)$$

so

$$k\left(\delta \sum_{j=1}^{n} B(t,T_j)\right) = B(t,T) - B(t,T_n)$$

and

$$k \equiv k(t,T,n) = \left(B(t,T) - B(t,T_n)\right)\left(\delta \sum_{j=1}^{n} B(t,T_j)\right)^{-1} \quad (20)$$

A swap is the forward (start payer) swap with $t = T$ and the (forward) swap rate, $k(T,T,n)$, is equal to

$$k(T,T,n) = \left(B(T,T) - B(T,T_n)\right)\left(\delta \sum_{j=1}^{n} B(T,T_j)\right)^{-1} \quad (21)$$

where $B(T,T) = 1$ of course and T corresponds to T_0 in our previous notation. So the forward swap rate is a special case of $k(t,T,n)$, when $t = T$. We left a few things unsaid in our definition. It is implicit that $k(t,T,n)$ is a function of the length of the swap — well, at least we cannot assume that it is independent of the time periods at this stage. We observe that the value of a swap (for us the payer swap settled in arrears) is a monotonic decreasing function of k, the fixed rate the parties agree for the contract. If we consider a single period swap with $T = T_0$ then

$$k(t,T_0,1) = \frac{B(t,T_0) - B(t,T_1)}{\delta B(t,T_1)}.$$

This coincides with the forward Libor rate over the time period $[T_0, T_1]$.

Apparently if one uses futures rates to determine swap rates then it can lead to arbitrage opportunities (Burghardt and Hoskins, 1995).

Swaptions

A payer swaption with strike rate k is the right but not the obligation to take up, at time T, the forward payer swap with fixed rate k, (settled in arrears). A market swap is one whose current value is zero, equivalently whose fixed rate is exactly the current swap rate. If the value, $FS_t(k)$, of our swap is non-negative at time T then, because both the market swap and our swap with rate k, have
the same floating payments to the holder, and differ

only in their fixed payments, it must be that k is less than (or equal to) the swap rate at time T (recall a swap is a monotone decreasing function of its fixed rate). It follows that the swap with the rate k is more favourable than one with the current swap rate.

For each $\omega \in \Omega$ for which $FS_T(k)(\omega) > 0$ one would exercise the swaption. Presumably one could sell your interest in this swap arrangement immediately thereby realising the payoff $FS_T(k)(\omega)$. Of course all of this won't work if $FS_T(k)(\omega) \leq 0$. So we can regard the payoff of this swaption as being $FS_T(k)^+$. This is entirely consistent with exercising the swaption and retaining one's interest in the swap in the event that k is less than the current swap rate (at time T).

Accordingly the value of the payer swaption, $PS_t(k)$ will satisfy,

$$PS_t(k) = M_t^{\mathbb{P}}\left(\frac{B_t}{B_T} FS_T(k)^+\right).$$

From our formula for $FS_t(k)$ we can rewrite the expression for $PS_t(k)$:

(from 16) $$PS_t(k) = M_t^{\mathbb{P}}\left(\frac{B_t}{B_T}\left(M_T^{\mathbb{P}}\left(\sum_{j=1}^{n} \frac{B_T}{B_{T_j}}(L(T_j)-k)\delta\right)\right)^+\right).$$

From (14) remembering $T = T_0$,

$$PS_t(k) = M_t^{\mathbb{P}}\left(\frac{B_t}{B_T}\left(\sum_{j=1}^{n} B(T,T_{j-1}) - \bar{\delta}\, B(T,T_j)\right)^+\right)$$

$$= M_t^{\mathbb{P}}\left(\frac{B_t}{B_T}\left(1 - \sum_{j=1}^{n} \delta_j B(T,T_j)\right)^+\right)$$

where $\delta_j = k\delta$ for $j < n$ and $\delta_n = 1 + k\delta$. This allows us to see the payer swaption as an option on a coupon bearing bond:

Consider a European call option on a bond which pays coupons $c_1, c_2, \dots, c_m$ at dates $T_1 \leq T_2 \leq \dots \leq T_m \leq T$ = expiry

The payoff of the option is (with strike K)

$$\left(\sum_{j=1}^{m} C_j B(T, T_j) - K \right)^+$$

and a put looks like,

$$\left(K - \sum_{j=1}^{m} C_j B(T, T_j) \right)^+ .$$

So our payer swaption can be seen as a put option struck at 1 on a coupon bond with coupons δ_j at times T_j. (The notional principle is 1).

Because for a random variable, X, we have $(-X)^+ = X^-$ then, writing,

$$RS_t(k) = M_t^{\mathbb{P}} \left(\frac{B_t}{B_T} \left(-FS_T(k) \right)^+ \right)$$

and noting this exactly an option on a swap in which the role of the "payer" is reversed (i.e. one receives fixed and pays floating) — such swaps are called receiver swaps — then

$$PS_t(k) + RS_t(k) = M_t^{\mathbb{P}_t} \left(\frac{B_t}{B_T} FS_T(k) \right)$$

$$= FS_t(k)$$

So, "payer swaption plus receiver swaption equals payer swap"

<u>Another equivalence:</u>

Consider a contract on a notional principle of $\frac{1}{k(T,T,n)}$ as follows. One receives the current swap rate and pays at the fixed rate k. So over a time period of length δ the net cash flow is

$$(k(T_i,T_i,n) - k)\delta \; .$$

This occurs over time periods, $[T_0,T_1]$, $[T_1,T_2]$, ..., $[T_{n-1},T_n]$. Imagine this quantity invested in $B(T_i)$ for the period $[T_i, T_n]$. This buys

$$\frac{(k(T_i,T_i,n)-k)\delta}{B(T_i)}$$

of bonds which has time T_n value

$$\frac{(k(T_i,T_i,n)-k)\delta}{B(T_i)} B(T_n)$$

adding it all up, the payoff from this arrangement is (at time T_n)

$$ir_T(k) = \sum_{J=1}^{n} (k(T_i,T_i,n)-k)\delta \frac{B(T_n)}{B(T_i)} \; .$$

Since we are in a complete market the time t value of this arrangement is

$$ir_t(k) = M_t^{\mathbb{P}}\left(\sum_{J=1}^{n} (k(T_i,T_i,n)-k)\delta \frac{B_t}{B(T_n)} \frac{B(T_n)}{B(T_i)} \right) .$$

Now,

$$k(T,T,n) = \frac{1 - B(T,T_n)}{\delta \sum_{j=1}^{n} B(T,T_j)}$$

so

$$\left(k(T,T,n) - k\right)\delta = \left(\frac{1 - B(T,T_n)}{\delta \sum_{j=1}^{n} B(T,T_j)} - k\right)\delta$$

$$= \left(\frac{1 - B(T,T_n) - k\delta \sum_{j=1}^{n} B(T,T_j)}{\sum_{j=1}^{n} B(T,T_j)}\right).$$

But also,

$$M_t^{\mathbb{P}}\left((k(T,T,n)-k)\delta\, M_T^{\mathbb{P}}\left(B_T/B_{T_j}\right) \right) = M_t^{\mathbb{P}}\left((k(T,T,n)-k)\delta\, B(T,T_j)\right)$$

(equation 13) so

$$M_t^{\mathbb{P}}\left(\sum_{j=1}^{n} (k(T,T,n)-k)\delta \frac{B_t}{B_{T_j}}\right) = M_t^{\mathbb{P}}\left(\sum_{j=1}^{n}(k(T,T,n)-k)\delta\, M_T^{\mathbb{P}}\left(\frac{B_T}{B_{T_j}}\right)\frac{B_t}{B_T}\right)$$

$$= M_t^{\mathbb{P}}\left((k(T,T,n)-k)\delta \cdot \sum_{j=1}^{n} B(T,T_j) \cdot \frac{B_t}{B_T} \right)$$

using the work above

$$= M_t^{\mathbb{P}}\left(\left(1 - B(T,T_n) - k\delta \sum_{j=1}^{n} B(T,T_j)\right)\frac{B_t}{B_T}\right)$$

or

$$= M_t^{\mathbb{P}}\left(\left(1 - \sum_{j=1}^{n} \delta_j B(T,T_j)\right)\frac{B_t}{B_T}\right)$$

where $\delta_j = k\delta$ if $j \leq n-1$, $\delta_n = 1+k\delta$. From equation (14)

$$1 - \sum_{j=1}^{n} \delta_j B(T,T_j) = FS_T(k)$$

So that

$$ir_t(k) = M_t^{\mathbb{P}}\left(FS_T(k)\, B_t / B_T \right) .$$

Which shows that the cash flows of this arrangem are identical with the forward payer swap settled in arrears.