The Lebesgue Integral : Introduction

The Riemann Integral, $\int_a^b f(x)\,dx$, defined in 1P1 serves most of the requirements of elementary analysis. But it does not meet all the requirements of advanced analysis. An extension of the Riemann integral, called the Lebesgue integral, permits a more general class of integrands, treats bounded and unbounded functions at once, and allows the intervals $[a,b]$ to be replaced by a far wider class of sets.

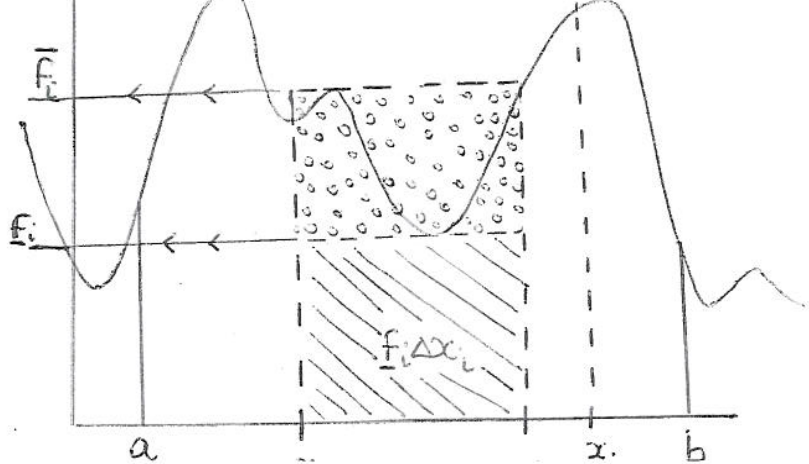
A comparison of the Riemann and Lebesgue approaches to integration will be helpful. Recall that with the Riemann integral we subdivide the interval $[a,b]$, i.e. the doma of f, and form sums:

$$\text{(lower sum)} \quad \underline{S} = \sum_i \underline{f}_i \,\Delta x_i \qquad \overline{S} = \sum_i \overline{f}_i \,\Delta x_i \quad \text{(upper sum)}$$

where $a = x_0 < x_1 < x_2 < \dots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$, $\underline{f}_i = \inf\limits_{x_{i-1} \leq x \leq x_i} f(x)$ and $\overline{f}_i = \sup\limits_{x_{i-1} \leq x \leq x_i} f(x)$. We call $\{x_i : 0 \leq i \leq n\}$ a partition of $[a,b]$.

Reminder

$\underline{f}_i \Delta x_i$ contributes the shaded area to $\underline{S}$ and $\overline{f}_i \Delta x_i$ contributes the shaded area plus the dotted area to $\overline{S}$.



Our function, f, will be integrable in the Riemann sense if there is a sequence of partitions of $[a,b]$ (which subdivide the interval into smaller and smaller subdivisions) for which the numbers $\overline{S} - \underline{S}$ tend to zero. Now

$$\overline{S} - \underline{S} = \sum_i (\overline{f}_i - \underline{f}_i)\Delta x_i$$

If f is cts on $[a,b]$ then it is unif cts · explain + picture, so $\overline{S}-\underline{S}$ gets small...

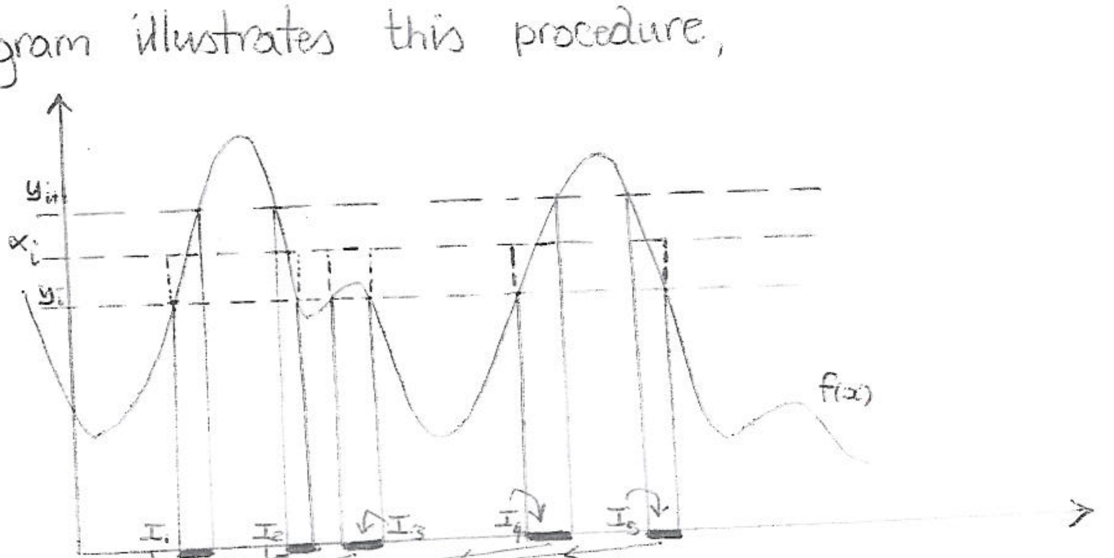
If f has only finitely many points of discontinuity then it is (more or less!) clear that breaking up $[a,b]$ into smaller and smaller subdivisions will make $\overline{f}_i - \underline{f}_i$ smaller and smaller (bar a few exceptional points) and the continued refinement will make $\overline{S} - \underline{S}$ tend to zero. But what if f has a discontinuity in every subinterval of $[a,b]$?① with which has positive length (Such functions are easily constructed). We cannot be sure that $\overline{f}_i - \underline{f}_i$ will get small as we take values of x closer together.

① Ex: Construct such a function.

Lebesgues answer to this problem was to change the way that the integral (area under the graph) is approximated. To get round the problem that values of $f(x)$ may differ wildly even as we take values of x closer together Lebesgue proposed that we approximate the integral not by dividing up the domain of f but by dividing up the range of f.

The diagram illustrates this procedure,

Subdivide the range with $y_0 < y_1 < \ldots < y_n$



$$E_i = \{ x : y_i \leq f(x) \leq y_{i+1} \}$$

We take the numbers y_i with $0 < y_{i+1} - y_i < \epsilon$ for each i. (By choosing ϵ "small" we collect together values of $f(x)$ that are "close".)

An approximation to the integral is given by

$$S' = \sum_i \alpha_i \, m(E_i)$$

where $\alpha_i \in (y_i, y_{i+1})$ and $m(E_i)$ is the sum of the lengths of the intervals $I_1 \dots I_5$. S' is just the sum of the areas of the rectangles of hieght α_i erected on $I_1, \dots I_5$. †

A moments thought will convince you that if f is sufficiently "beserk" then the set E_i may have a very complicated form. How can we measure the length of such a set ?.

The first section of our course will answer this question.

<u>Remark</u> R-int is like adding quantities in the order given (by x). L-int is like collecting together like terms, adding these and then summing the result. (Example with finite sequence of coins).

END 1st LECTURE

† You might say $\alpha_i m(E_i)$ is the area of the rectangle with hieght α_i based on E_i.

1. Lebesgue Measure on $\mathbb{R}$.

The strategy is this; define the measure of "simple" sets (in a simple and obvious way) then "construct" ~~approximate~~ more complicated sets with simple sets and use the measures of the simple sets to define the measure of the complicated set. For this strategy to work we need a working knowledge of how simple sets combine and how the measure behaves when we combine simple sets. That's our first task.

1.1 Definition

(a) An interval is a (bounded) subset of $\mathbb{R}$ with one of the following forms; let $a, b \in \mathbb{R}$ $a \leq b$

$$[a,b] \quad [a,b), \quad (a,b], \quad (a,b).$$

(Note that $\emptyset$ and $\{a\}$, $a \in \mathbb{R}$ are therefore intervals!)

(b) A basic subset of $\mathbb{R}$ is a union of finitely many disjoint intervals. (So, in particular, an interval is a basic set)

(c) For an interval, I, $\mu(I)$ will denote its measure and the following gives its value;

$$\mu[a,b] = \mu(a,b] = \mu[a,b) = \mu(a,b) = b - a$$

Hence $\mu\emptyset = \mu\{a\} = 0$. So the empty set and "single points" have measure zero.

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Given a basic set E we know that there are intervals $I_1 \dots I_k$ (say) such that $E = \bigcup_{n=1}^{k} I_n$. These intervals may of course overlap but we can write E as a union of disjoint intervals as follows, let

$$J_1 = I_1$$
$$J_2 = I_2 \setminus I_1$$
$$J_3 = I_3 \setminus I_2 \setminus I_1$$
$$\vdots$$
$$J_k = I_k \setminus I_{n-1} \setminus I_{n-2} \setminus \dots \setminus I_1$$

It is not difficult to prove that the J_i's are unions of disjoint intervals and since

$$\bigcup_{i=1}^{r} J_i = I_1 \cup (I_2 \setminus I_1) \cup (I_3 \setminus I_2 \setminus I_1) \cup \dots$$
$$= I_1 \cup \dots \cup I_r$$

$*$ see exercises

that $\bigcup_{i=1}^{k} J_i = \bigcup_{n=1}^{k} I_n = E$. This shows us something else too. A given basic set E may be written in more than one way as a union of intervals. This is still true even if we stipulate that the intervals be disjoint!;

$$[0,1] = [0,\tfrac{1}{2}) \cup [\tfrac{1}{2},1] = [0,\tfrac{1}{4}) \cup [\tfrac{1}{4},\tfrac{1}{2}) \cup [\tfrac{1}{2},\tfrac{3}{4}) \cup [\tfrac{3}{4},1]$$

This has a consequence for the following definition.

1.2 Definition

Let E be a basic set and suppose $E = \bigcup_{i=1}^{k} J_i$ where $J_i \cap J_\ell = \emptyset$ if $i \neq \ell$ and each J_i is an interval.

We define $\mu E \overset{\text{def}}{=} \sum_{i=1}^{k} \mu J_i$.

The following shows that μE is <u>well defined</u>

<u>Explain well defined</u>

1.3 Proposition

Let E be a basic set and suppose $E = \bigcup_{i=1}^{k} J_i = \bigcup_{j=1}^{\ell} I_j$ where each of $\{I_j\}$ and $\{J_i\}$ is a family of mutually disjoint intervals

then $\sum_{i=1}^{k} \mu J_i = \sum_{j=1}^{\ell} \mu I_j$.

<u>Proof</u> Let $N_{ij} = J_i \cap I_j$ $\quad 1 \leq i \leq k,\ 1 \leq j \leq \ell$.

Then each N_{ij} is an interval and $\{N_{ij} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ is a disjoint collection (ie. $N_{ij} \cap N_{sr} = \emptyset$ whenever $i \neq s$ <u>or</u> $j \neq r$). Further,

$$I_j = \bigcup_{i=1}^{k} N_{ij} \overset{\text{because}}{=} (I_j \cap J_1) \cup (I_j \cap J_2) \cup \ldots$$
$$= I_j \cap \left(\bigcup_{i=1}^{k} J_i\right) \text{ (de Morgans Laws)}$$
$$= I_j \cap E$$
$$= I_j$$

and similarly $J_i = \bigcup_{j=1}^{\ell} N_{ij}$. But I_j is an interval say with endpoints $a, b, a \leq b$, and $\bigcup_{i=1}^{k} N_{ij}$ is I_j expressed as a disjoint union of intervals, so if N_{ij} has end points a_i, b_i with $a_i \leq b_i$ there must be i_0, $1 \leq i_0 \leq k$ for which a_{i_0} = left end point of I_j

(if not argue to get a contradiction of relation $I_j = \bigcup_{i=1}^{k} N_{ij}$). Now consider b_{i_0}. It must be that b_{i_0} is the <u>left</u> end point of some interval $N_{i_1 j}$ for if not $(b_{i_0}, \min_{i \neq i_0} a_i)$ is not part of I_j so I_j has a "gap" in it — but it cannot or an interval is <u>connected</u>. Carrying on in this way we get an arrangement the N_{ij}'s; $N_{i_0 j}, N_{i_1 j}, N_{i_2 j}, \dots, N_{i_k j}$ such that with end points $a_{i_0}, b_{i_0}, a_{i_1}, b_{i_1}, \dots, a_{i_k}, b_{i_k}$

$$a = a_{i_0} \leq b_{i_0} = a_{i_1} \leq b_{i_1} = a_{i_2} \leq \dots = a_{i_k} \leq b_{i_k} = b .$$

$$\text{Now } \mu I_j = b - a = b_{i_k} - a_{i_0} = b_{i_k} - a_{i_k} + b_{i_{k-1}} - a_{i_{k-1}} + b_{i_{k-2}} - \dots + b_{i_0} - a$$

(remember $b_{i_r} = a_{i_{r+1}}$) $= \mu N_{i_k j} + \mu N_{i_{k-1} j} + \dots + \mu N_{i_0 j}$

$$= \sum_{i=1}^{k} \mu N_{ij} \quad \text{because } N_{i_r j}\text{'s are just}$$

a rearrangement of the N_{ij}'s. To finish the note that

$$\sum_{j=1}^{\ell} \mu I_j = \sum_{j=1}^{\ell} \sum_{i=1}^{k} \mu N_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \mu N_{ij}$$

and that we can use <u>precisely</u> the argument above to conclude that $\sum_{i=1}^{k} \sum_{j=1}^{\ell} \mu N_{ij} = \sum_{i=1}^{k} \mu J_i$. $\square$

END LECTURE 2.

<u>Properties of Basic Sets and the Measure</u>

<u>1.4 Proposition</u>

Let $\mathcal{B} = \{E : E \text{ is a basic set}\}$. If $E, F \in \mathcal{B}$ then

(i) $E \cup F \in \mathcal{B}$

(ii) $E \cap F \in \mathcal{B}$

DISPLAY

Rem: Additivity

(iii) $E \setminus F \in \mathcal{B}$

(iv) If $E \cap F = \emptyset$ then $\mu E \cup F = \mu E + \mu F$. And hence by induction $\mu(E_1 \cup E_2 \cup \dots \cup E_k) = \sum_{i=1}^{k} \mu E_i$ for disjoint $E_i \in \mathcal{B}$, $k \in \mathbb{N}$.

(v) $E, F \in \mathcal{B}$, $E \subseteq F$ then $\mu E \leq \mu F$. Rem Monotonicity.

Proof

(i), (ii), (iii) (v) are exercises

(iv) Let $E = \bigcup_{j=1}^{k} I_j$ $F = \bigcup_{i=1}^{\ell} J_i$ $\{I_j\}, \{J_i\}$ each a family of pairwise disjoint sets. Since E and F are disjoint $\{I_1, \dots I_k, J_1, \dots, J_\ell\}$ is a disjoint family whose union is $E \cup F$.

Hence $\mu(E \cup F) = \sum_{j=1}^{k} \mu I_j + \sum_{i=1}^{\ell} \mu J_i = \mu E + \mu F$. □

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1.5 Remark

If $E, F \in \mathcal{B}$ are not necessarily disjoint how does $\mu(E \cup F)$ compare with $\mu E + \mu F$?. Well

$E \cup F = E \setminus F \cup E \cap F \cup F \setminus E$ and $\{E \setminus F, E \cap F, F \setminus E\}$ is a disjoint family hence (applying 1.4 (iv) twice)

$\mu(E \cup F) = \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E)$. Observe that

$E = E \setminus F \cup E \cap F$ and $\{E \setminus F, E \cap F\}$ is a disjoint family

So $\mu E = \mu(E \setminus F) + \mu(E \cap F)$, so

$$\mu(E \cup F) = \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E) + \mu(E \cap F) - \mu(E \cap F)$$

$$= \mu E + \mu F - \mu(E \cap F)$$

$$\leq \mu E + \mu F \quad \text{because } \mu G \geq 0 \text{ for } G \in \mathcal{B}.$$

We call 1.4 (iv) additivity of μ. The relation above might be paraphrased "μ is subadditive across unions".

Our last result of this section shows that a stronger version of 1·4 (iv) holds. It states that μ is <u>countably</u> additive on $\mathcal{B}$, more precisely

6 Theorem $\square$

(i) Let (E_n) be a sequence of basic sets. Let $E \in \mathcal{B}$ be such that; $E \subseteq \bigcup_{n=1}^{\infty} E_n$ then

$$\mu E \leq \sum_{n=1}^{\infty} \mu E_n$$

(ii) If in addition to the hypotheses of (i) we have $E_m \cap E_n = \emptyset$ for $m \neq n$ and $E = \bigcup_{n=1}^{\infty} E_n$ then $\mu E = \sum_{n=1}^{\infty} \mu E_n$.

Proof

(i) If $\sum \mu E_n$ diverges then the result is trivially true because μE is always finite. So assume that $\sum \mu E_n$ converges. Since $E \in \mathcal{B}$ there are intervals $I_1, \dots, I_k$ say for some $k \in \mathbb{N}$ such that $E = \bigcup_{r=1}^{k} I_r$, $I_i \cap I_j = \emptyset$ for $i \neq j$. Now let

$$E^* = E \cup \{x : x \text{ is an } \overset{\text{(end)}}{\text{accumulation}} \text{ point of } I_i \text{ for some } i.\}$$

This is merely adding the endpoints of the I_i's if they are absent thus ensuring that E^* is closed and bounded and hence compact. Now $E^* \in \mathcal{B}$ for it was obtained from E by adding finitely many (single) points, since points have measure zero it follows (from 1·4(iv)) that $\mu E^* = \mu E$.

Now let us enlarge the collection of basic sets E_n to include the singletons containing the end points that we have added. This gives us a collection $\{E_n' : n \in \mathbb{N}\}$ where, $E_i' = \{x_i\}$ for $1 \leq i \leq 2k$ and $E_{2k+1}' = E_1$, $E_{2k+2} = E_2, \ldots, E_{2k+r} = E_r$, and so on.

Since $\mu\{x_i\} = 0 \;\; \forall i$, we have that

$$\sum_{n=1}^{\infty} \mu E_n = \sum_{n=1}^{\infty} \mu E_n' ,$$

and clearly $$E^{(t)} \subseteq \bigcup_{n=1}^{\infty} E_n' .$$

Now as each E_n' is bounded then for each $\epsilon > 0$ there is an open set F_n with $F_n \supseteq E_n'$ and $\mu F_n < \mu E_n' + \epsilon/2^n$.

To see this let J be any interval and $\delta > 0$, the interval $K = (\inf J - \frac{\delta}{2}, \sup J + \frac{\delta}{2})$ is open and $\mu K = \mu J + \delta$. If $E_n' = \bigcup_{r=1}^{k} J_r$ (J_r intervals) with $J_i \cap J_j = \emptyset$ for $i \neq j$ let J_r' be such that $J_r' \supseteq J_r$, J_r' open and $\mu J_r' = \mu J_r + \frac{\epsilon}{2^{n+1} k}$, $1 \le r \le k$

Then $F_n = \bigcup_{r=1}^{k} J_r'$ is open $F_n \supseteq E_n'$ and

$$\mu F_n = \mu\left(\bigcup_{r=1}^{k} J_r'\right) \leq \sum_{r=1}^{k} \mu J_r' \quad \text{by subadditivity}$$

$$= \sum_{r=1}^{k} \left(\mu J_r + \frac{\epsilon}{2^{n+1} k}\right) \quad \text{by choice of } J_r'$$

$$= \left(\sum_{r=1}^{k} \mu J_r\right) + \frac{\epsilon}{2^{n+1}}$$

$$< \mu E_n' + \frac{\epsilon}{2^n} \quad \text{because}$$

Hence $\bigcup_{n=1}^{\infty} F_n \supseteq \bigcup_{n=1}^{\infty} E_n' \supseteq E^*$. So (F_n) is an open cover of E^*, E^* is compact and so there is a finite subcover, say $F_{n_1}, \ldots, F_{n_m}$. So that

$$E^* \subseteq \bigcup_{r=1}^{m} F_{n_r}$$

and hence

$$\mu E^* \leq \sum_{r=1}^{m} \mu F_{n_r} \quad \text{(subadd')}$$

$$\leq \sum_{n=1}^{\infty} \mu F_n$$

and so

$$\mu E^* \leq \sum_{n=1}^{\infty} \mu F_n \leq \sum_{n=1}^{\infty} (\mu E_n' + \epsilon/2^n)$$

$$= \left(\sum_{n=1}^{\infty} \mu E_n'\right) + \sum_{n=1}^{\infty} \epsilon/2^n$$

$$= \sum_{n=1}^{\infty} \mu E_n + \epsilon$$

So $\mu E = \mu E^* \leq \sum_{n=1}^{\infty} \mu E_n + \epsilon$ for any $\epsilon > 0$. It follows that $\mu E \leq \sum_{n=1}^{\infty} \mu E_n$.

(ii) If $E = \bigcup_{n=1}^{\infty} E_n$ then $\forall m \in \mathbb{N}$ $E \supseteq \bigcup_{n=1}^{m} E_n$ and hence $\mu E \geq \mu\left(\bigcup_{n=1}^{m} E_n\right) = \sum_{n=1}^{m} \mu E_n$. So $\sum_{n=1}^{\infty} \mu E_n \leq \mu E$.

Combining this relation with that just proved in (i) gives the result. $\square$.

END 3 OR AT * BELOW

The property that μ displays in 1.6(ii) is called countably additivity of μ on $\mathcal{B}$. It is a vital property.

Summary

We have identified a class ($\mathcal{B}$) of "simple" sets closed under the formation of unions, intersections and difference. We have defined the measure of such sets. The measure is a non-negative function on $\mathcal{B}$ which is monotone, additive (on disjoint unions), subadditive on arbitrary unions, countably additive on $\mathcal{B}$

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