In abstracting our approach to integration theory we find that theorem 7:12 is most useful. It is often the case that a good theorem becomes a definition! .

11·1 Definition

Let (Ω, Σ, μ) be a σ-finite measure space. Let $f : \Omega \to \bar{\mathbb{R}}$. Then f is called a measurable transformation (w.r.t. Σ) $\iff \{f \leq c\} \in \Sigma \; \forall c \in \bar{\mathbb{R}}$. Note that this does <u>not</u> depend upon μ. One can prove

11 : 2 Theorem

Let (Ω, Σ, μ) be a σ-finite measure space. Let f, g be measurable transformations and $c \in \mathbb{R}$. Then

(i) $f + cg$ is measurable

(ii) $|f|$ is measurable

(iii) $f.g$ is measurable

(iv) If (f_n) is a sequence of measurable functions and $f_n \uparrow f$ then f is measurable. ($f_n \downarrow f$ works too).

(v) If $f_n \longrightarrow f$ pointwise and each f_n is measurable then so is f.

(vi) Let $A \subseteq \Omega$ and $\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in \Omega \setminus A \end{cases}$. Then

$$A \in \Sigma \iff \chi_A \text{ is a measurable function.}$$

Proof Try (i), (vi).

The construction of the integral can now proceed with simple functions replacing step functions. A function f is simple if it is of the form

$$f(\omega) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(\omega) \quad \text{where} \quad E_i \in \Sigma \text{ and } \mu E_i < \infty .$$

Instead of using upper and lower functions a slightly more economic development is obtained by using converge in measure : A sequence (f_n) of measurable functions converges in μ-measure to the (measurable !) function f if $\forall \epsilon > 0$ $\lim_{n \to \infty} \mu \{\omega : |f(\omega) - f_n(\omega)| > \epsilon\} = 0$

The integral of a simple function $f = \sum_i \alpha_i \chi_{E_i}$ is given by

$$\int_\Omega f d\mu = \sum_i \alpha_i \mu E_i$$

If f is measurable, μ-a.e. finite, and $\exists$ simple functions (f_n) :

1. (f_n) converges to f in μ measure
2. $\int_\Omega |f_n - f_m| d\mu \to 0$ as $m,n \to \infty$

then we say f is μ-integrable. In this case

$\lim_{n\to\infty} \int_\Omega f_n \, d\mu$ exists and we define $\int_\Omega f \, d\mu = \lim_n \int_\Omega f_n \, d\mu$.

You should make a note of the following theorems.

11:3 Theorem

Let (f_n) be a sequence of integrable functions converging to f μ-a.e. (or in μ-measure!). Let g be integrable and $\forall n \; |f_n(\omega)| \leq |g(\omega)|$ μ-a.e. Then f is integrable and $\lim_n \int_\Omega |f - f_n| \, d\mu \longrightarrow 0$. Moreover

$$\lim_n \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu .$$

Proof Very similar to 7·6

11:4 Theorem (Monotone Convergence)

Let f_n be integrable and $f_n \uparrow$ as $n \uparrow$ and $\sup_n \int_\Omega f_n \, d\mu < \infty$. Then $\lim_n f_n(\omega) = f(\omega)$ is finite μ-a.e. and f is integrable. Moreover $\int_\Omega f_n \, d\mu \uparrow \int_\Omega f \, d\mu$.

Proof Similar to 5·7.