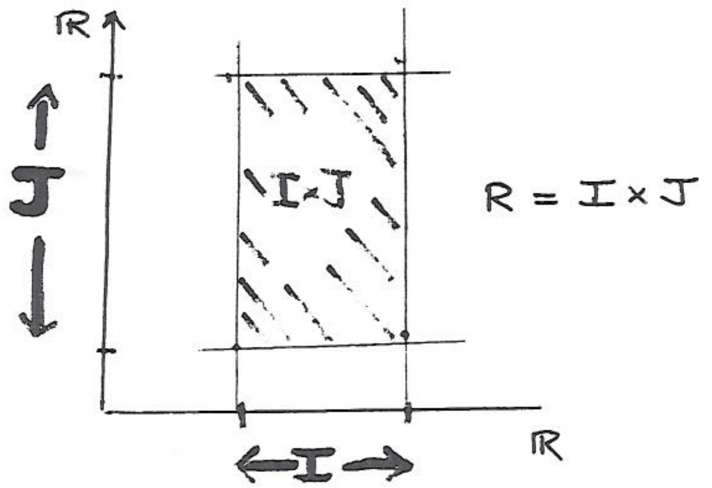
12: Lebesgue Measure on $\mathbb{R}^2$ and Fubini's Theorem

We are going to use some of the results of §10 and §11 ~~to the situation~~ in $\mathbb{R}^2$. How do we measure the size of sets in $\mathbb{R}^2$?. Well, by their area!. Start off with rectangles i.e. the Cartesian product of intervals in $\mathbb{R}$



As before —

A basic subset of $\mathbb{R}^2$ is a finite union of rectangles.

An outer set in $\mathbb{R}^2$ is the union of a sequence of basic subsets.

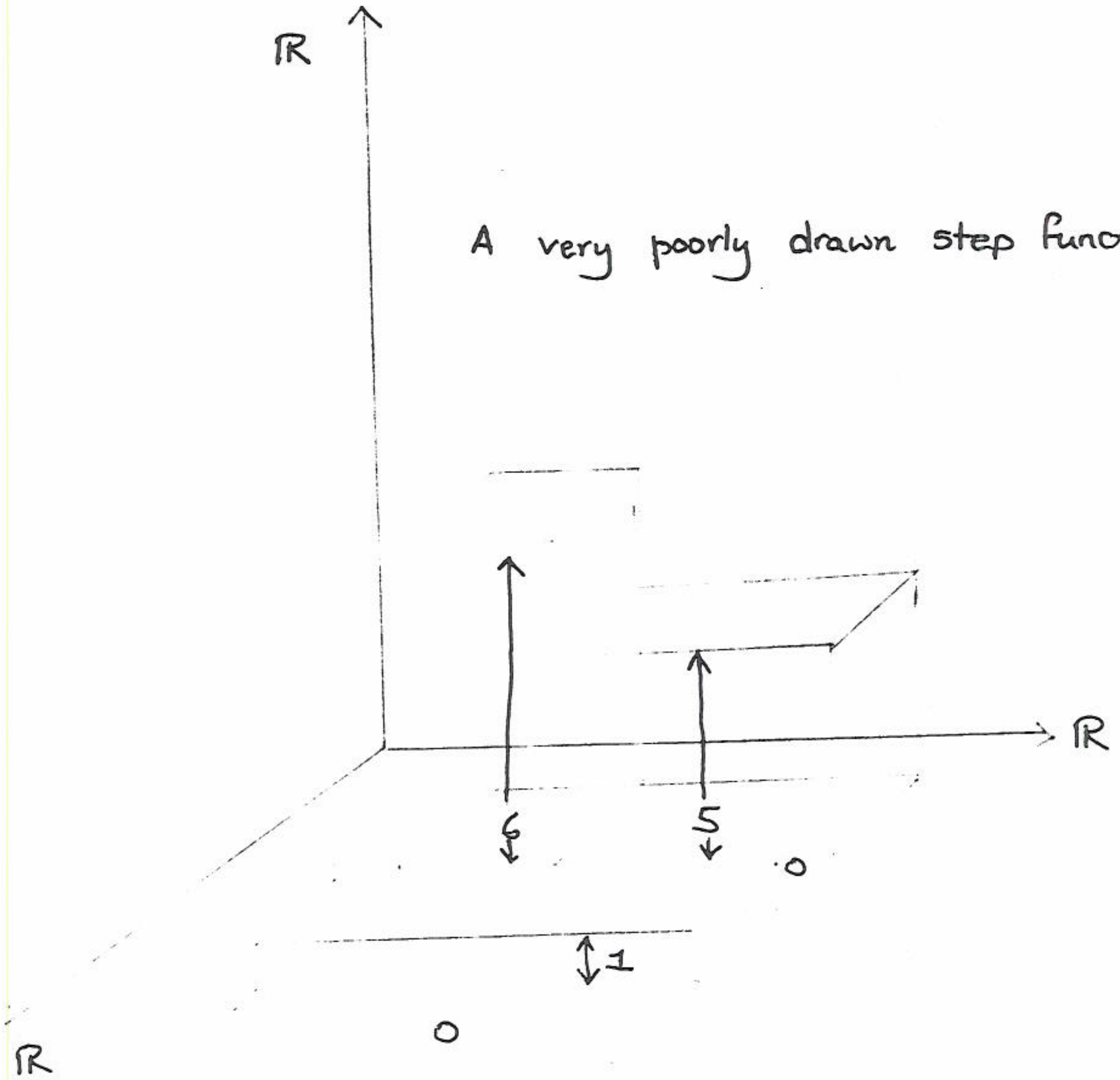
An inner set in $\mathbb{R}^2$ is a bounded set whose complement is outer.

If a rectangle $R = I \times J$ and μ is Lebesgue measure on $\mathbb{R}$ then the Lebesgue measure of R in $\mathbb{R}^2$ is just

$\lambda(R) = \mu I \cdot \mu J$ as you would expect. From this the measure of basic, outer, and inner sets may be defined and finally a set $E \subseteq \mathbb{R}^2$ is integrable if $\forall \varepsilon > 0$ $\exists$ G outer and H inner with $H \subseteq E \subseteq G$ and $\lambda(G \setminus H) < \varepsilon$

. The definition of measurable set is much as before; $\forall n$ $E \cap ([-n,n] \times [-n,n])$ is integrable. Let $\mathcal{M}(\mathbb{R}^2)$ denote the measurable subsets of $\mathbb{R}^2$.

Integrable functions $f : \mathbb{R}^2 \to \mathbb{R}$ can be defined just as in §4, 4:13(iv) via upper and lower functions $(\mathbb{R}^2 \to \mathbb{R})$
Integrable functions defined this way coincide with those defined as * (see ov
A step function in $\mathbb{R}^2$ now looks rather like Manhatten, New York – but without the lights.



A very poorly drawn step function

The result we shall prove is,

12.1 Theorem (Fubini)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be integrable. Then for μ-almost all $y \in \mathbb{R}$, the function $F_y(x)$ given by

$$x \longmapsto f(x, y)$$

is integrable $\mathbb{R} \to \mathbb{R}$. Moreover the function

* in §10 and §11 for the measure space $(\mathbb{R}^2, \mathfrak{M}(\mathbb{R}^2), \lambda)$

$$y \longmapsto \int_{\mathbb{R}} F_y(x)\,dx = \int_{\mathbb{R}} f(x,y)\,dx \quad \text{is integrable, } \mathbb{R} \to \mathbb{R},$$

and $$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^2} f \, d\lambda \qquad \left(\begin{array}{l} \lambda \text{ is Lebesgue} \\ \text{measure on } \mathbb{R}^2 \end{array} \right)$$

There follows immediately the Corollary,

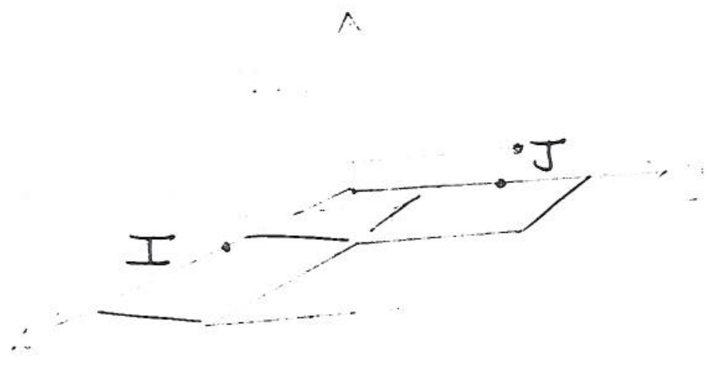
12.2 Corollary

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dy \right) dx = \int_{\mathbb{R}^2} f \, d\lambda$$

Proof

Suppose $f = \chi_R$ where $R = I \times J$ is a rectangle.

Picture the function f,



Fix y then $x \longmapsto f(x,y)$ is just $\begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \in \mathbb{R} \setminus I \end{cases}$

that is $\begin{cases} \forall y \in J & F_y(x) = \chi_I(x) \\ y \in \mathbb{R}\setminus J & F_y(x) = 0 \end{cases}$. Now χ_I is integrable

and $\begin{cases} \int_{\mathbb{R}} F_y(x)\,dx = \int_{\mathbb{R}} \chi_I \, d\mu = \mu I & y \in J, \text{ (μ, remember, is Lebesgue measure on $\mathbb{R}$)} \\ \int_{\mathbb{R}} F_y(x)\,dx = 0 & y \in \mathbb{R}\setminus J, \end{cases}$

Now $y \longmapsto \int_{\mathbb{R}} F_y(x)\,dx$ is just the function $\begin{cases} \mu I & \text{if } y \in J \\ 0 & \text{if } y \notin J \end{cases}$

ie. $\mu I \chi_J$ so this is certainly μ-integrable and

$$\int_{\mathbb{R}} \mu I \chi_J \, d\mu = \mu I \int_{\mathbb{R}} \chi_J \, d\mu = \mu I \cdot \mu J = \lambda(I \times J)$$
$$= \lambda(R).$$

$$\text{So} \quad \lambda(R) \equiv \int_{\mathbb{R}^2} \chi_R \, d\lambda \equiv \int_{\mathbb{R}^2} f \, d\lambda = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy$$

In other words the result is true for the characteristic function of a rectangle.

A step function is a linear combination of characteristic functions of <u>disjoint</u> rectangles. So the result for a step function on $\mathbb{R}^2$ follows by linearity of the integral | additivity of λ.

Now we consider an <u>integrable upper function</u> ie. an f which is the limit (possibly-∞ô̇t) of an increasing sequence of step functions. <u>Warning this is where it starts to get difficult</u>. We have $f_n(x,y)$ — step functions — and $f_n(x,y) \uparrow f(x,y)$ for λ-almost every $(x,y) \in \mathbb{R}^2$, and $\int_{\mathbb{R}^2} f\,d\lambda = \lim \int_{\mathbb{R}^2} f_n \, d\lambda$:

Let $S = \{(x,y) \in \mathbb{R}^2 : f_n(x,y) \not\uparrow f(x,y)\}$. Now the result we seek is true for a step function on $\mathbb{R}^2$ and so

$$\int_{\mathbb{R}^2} f_n \, d\lambda = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_n(x,y)\,dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_n(x,y)\,dy \right) dx$$

Let $g_n(x) = \int_{\mathbb{R}} f_n(x,y)\,dy \quad x \in \mathbb{R}$. Then g_n is a step function on $\mathbb{R}$ and $\int_{\mathbb{R}^2} f_n \, d\lambda = \int_{\mathbb{R}} g_n \, d\mu$. Since $f_{n+1} \geq f_n$ it follows that $g_{n+1} \geq g_n$ (why?).

Now suppose that $x_0 \in \mathbb{R}$ is such that $\{y : (x_0, y) \in S\}$ is a μ-null set (a null set in $\mathbb{R}$) and also $(g_n(x_0))$ converges. Then for μ-almost all y, $f_n(x_0,y) \uparrow f(x_0, y)$ and $\sup_n \int_{\mathbb{R}} f_n(x_0,y)\,dy < \infty$. By the Monotone convergence theorem $f(x_0,y) \in \mathcal{L}$ and $\int_{\mathbb{R}} f(x_0,y)\,dy = \lim_n g_n(x_0)$

Lets summarise this; If $x_0 \in \mathbb{R}$ is such that $S_{x_0} = \{y : (x_0,y) \in S\}$ is a μ-null set and $\sup_n \int_{\mathbb{R}} f_n(x_0,y)\,dy < \infty$ then $f(x_0,y)$ (as a function of y) is integrable and $\int_{\mathbb{R}} f(x_0,y)\,dy = \lim_n g_n(x_0)$.

O.K., Now suppose further that

for μ-almost every $x_0 \in \mathbb{R}$, S_{x_0} is a μ-null set and $\lim_n g_n(x_0)$ exists. ∞

Let $g(x_0) = \lim_n g_n(x_0) = \lim_n \int_{\mathbb{R}} f_n(x_0, y)\,dy$ for those (μ-almost all) x_0 for which S_{x_0} is a μ-null set. Then g is the μ-a.e. limit of the increasing sequence of step functions g_n and $\int_{\mathbb{R}} g_n\,d\mu = \int_{\mathbb{R}^2} f_n\,d\lambda$ and by hypothesis

$\infty > \sup_n \int_{\mathbb{R}^2} f_n\,d\lambda \geqslant \sup_n \int_{\mathbb{R}} g_n\,d\mu$. So g is an integrable upper function and

$$\int_{\mathbb{R}} g\,d\mu = \lim_n \int_{\mathbb{R}} g_n\,d\mu = \lim_n \int_{\mathbb{R}^2} f_n\,d\lambda \overset{\text{by def}^n}{=} \int_{\mathbb{R}^2} f\,d\lambda$$

Recalling just what $\lim_n g_n(x)$ <u>is</u> and using monotone convergence we get

$$\int_{\mathbb{R}^2} f\,d\lambda \underset{\text{above}}{\overset{\text{as}}{=}} \lim_n \int_{\mathbb{R}} g_n\,d\mu \overset{\text{M.C.Thm}}{=} \int_{\mathbb{R}} \left(\lim_n g_n\right) d\mu \underset{\lim_n g_n ?}{\overset{\text{what is}}{=}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dy \right)$$

This does it for upper functions. We do integrable functions later

First note that,

This proof is biult upon the as yet unproved assumption that,

" For μ-a.every x_0 , S_{x_0} is a μ-null set " .

But is this true ? , we know S is λ-null, does this tell us that S_{x_0} is μ-null for μ a.every x_0 ?

12·4 Lemma

Let $S \subseteq \mathbb{R}^2$ be a null set. $S_x = \{y : (x,y) \in S\}$. Then S_x is μ-null for μ-almost all $x \in \mathbb{R}$.

Proof

This uses a techical result the analogue of which is Lemma 4:17. More precisely if $S \subseteq \mathbb{R}^2$ is λ-null, $\epsilon > 0$ there is an upper function $h \geq 0$ with $h \restriction S = \infty$ and $\int_{\mathbb{R}^2} h \, d\lambda \leq \epsilon$. We can find therefore an increasing sequence of step functions $h_n \uparrow h$ with $\sup_n \int_{\mathbb{R}^2} h_n d\lambda \leq \epsilon$ and $h_n(x,y)$ divergento for each $(x,y) \in S$.

Let $g_n(x) = \int_{\mathbb{R}} h_n(x,y)\,dy$, (recall argument in 12·3) then g_n is a step function $(\mathbb{R} \to \mathbb{R})$ and

$$\int_{\mathbb{R}} g_n \, d\mu = \int_{\mathbb{R}^2} h_n \, d\lambda$$ by Fubini's Theorem for a step funct (which doesn't require this lemma!). Once again $g_n \uparrow$ and $\lim_n \int_{\mathbb{R}} g_n < \infty$. By theorem 4:11 $\{x : \lim_n g_n(x) \text{ diverges}\}$ is a μ-null set ($\lim_n g_n(x) = g(x)$ is an integrable upper f^n & hence is finite a.e.).

Hence $\{x : \left(\int_{\mathbb{R}} h_n(x,y)\,dy\right) \text{ diverges}\}$ is μ-null. Now (for fixed x_0) if $\left(\int_{\mathbb{R}} h_n(x_0,y)\,dy\right)$ converges then — again by 4:11

$h_n(x_0, y)$ converges for μ almost every y, hence

$S_{x_0} = \{ y : (h_n(x_0, y)) \text{ diverges} \}$ is μ-null, for μ almost all x_0.

$\square$

Final Part of the proof of Fubini's Theorem

Let F be integrable.

We have the result for upper functions and hence for lower functions now choose $\epsilon > 0$ and an upper function h with

$$\int_{\mathbb{R}^2} h \, d\lambda < \int_{\mathbb{R}^2} f d\lambda + \epsilon \;, \quad h \geq f \quad \forall (x,y)$$

Let $H(y) = \int_{\mathbb{R}} h(x,y) dx$ and $F^*(y) = \int^*_{\mathbb{R}} f(x,y) dx$

Since $h \geq f$, $F^*(y) \leq H(y) < \infty$ for almost every y

So
$$\int^*_{\mathbb{R}} F^*(y) \, dy \leq \int_{\mathbb{R}} H(y) \, dy \overset{\text{by}}{\underset{\text{Fubini}}{=}} \int_{\mathbb{R}^2} h \, d\lambda$$
$$< \int_{\mathbb{R}^2} f d\lambda + \epsilon \, .$$

Now consider a lower function $g \leq f \;\; \forall (x,y)$, let

$F_*(y) = \int_{*\mathbb{R}} f(x,y) dx \quad \forall y$ then because g is integrable

$F_*(y) > -\infty$ for almost all y and we can choose g so that $\int_{*\mathbb{R}} F_*(y) dy > \int_{\mathbb{R}^2} f d\lambda - \epsilon$. We always have

$F_*(y) \leq F^*(y)$ and it follows that

$$\int_{\mathbb{R}^2} f\,d\lambda - \epsilon \leqslant \int_* F_* \leqslant \int^* F_* \leqslant \int^* F^* < \int_{\mathbb{R}^2} f\,d\lambda + \epsilon$$

this uses the fact that $\int_* \leqslant \int^*$ and $U \leqslant V \Rightarrow \int^* u \leqslant \int^* v$ (similar for lower integrals). It follows that F_* and F^* are integrable and $\int_{\mathbb{R}} F_* d\mu = \int_{\mathbb{R}} F^* d\mu = \int_{\mathbb{R}^2} f\,d\lambda$. Since $F^* - F_* \geq 0$

and $\int_{\mathbb{R}} (F^* - F_*)\,d\mu = 0$ we have $F_* = F^*$ a.e. Let

$F = F_* = F^*$ a.e., then $F(y) = \int^*_{\mathbb{R}} f(x,y)\,dx = \int_{*\mathbb{R}} f(x,y)\,dx < \infty$

for a.every $y \in \mathbb{R}$, hence $f(\cdot, y)$ is integrable for almost all $y \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^2} f\,d\lambda \qquad \text{oh le !}$$

$\square$.

A TEST FOR INTEGRABILITY OF FUNCTIONS $f : \mathbb{R}^2 \to \mathbb{R}$.
(YOU SHOULD "EXAMINE" THE PROOF CLOSELY).

* Theorem (Hobson)

Suppose f is measurable on $\mathbb{R}^2$ and that at least one of the iterated integrals $\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x,y)|\,dx\right)dy$ or $\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x,y)|\,dy\right)dx$ exists. Then $f \in \mathcal{L}(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} f\,d\lambda = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,dx\right)dy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,dy\right)dx .$$

Proof

Suppose $\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x,y)|\,dx\right)dy$ exists. Define a sequence of step functions, $S_n(x,y) = \begin{cases} n & \text{if } |x| \le n,\ |y| \le n \\ 0 & \text{otherwise} \end{cases}$.

(picture CB !). Let $f_n(x,y) = \min\{S_n(x,y), |f(x,y)|\}$.

Then $0 \le f_n \le S_n$ and f_n is measurable because each of S_n and $|f|$ are (remember f measurable $\Rightarrow |f|$ measurable). But S_n is integrable, hence (by the Dominated Convergence Theorem) f_n is integrable. By Fubini's Theorem

$$\int_{\mathbb{R}^2} f_n\,d\lambda = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f_n(x,y)\,dx\right)dy \underset{0 \le f_n \le |f|}{\overset{\text{because}}{\le}} \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(x,y)|\,dx\right)dy$$

Hence $\sup_n \int_{\mathbb{R}^2} f_n \, d\lambda < \infty$ and so $\lim_n f_n$ is integrable by the Monotone Convergence Theorem. But $\lim_n f_n = |f|$

Hence $|f|$ is integrable and f is measurable, so f is integrable.