3 Finitely Measurable Sets

3.1 Definition

Let $E \subseteq \mathbb{R}$. We define the outer measure of E by

$$\mu^* E = \inf\{\mu F : F \supseteq E,\ F \text{ outer}\}$$

and the inner measure by

$$\mu_* E = \sup\{\mu G : G \subseteq E,\ G \text{ inner}\}$$

A set E is integrable or finitely measurable if $\mu_* E = \mu^* E$ and we write μE, the measure of E, for this common value.

Corollarys

(i) E is integrable with measure μE if and only if $\forall \epsilon > 0$ $\exists$ F outer and G inner with $G \subseteq E \subseteq F$ and $\mu F - \epsilon < \mu E < \mu G + \epsilon$.

(ii) E, is integrable with measure μE if and only if $\exists (G_n), (F_n)$ sequences of (respectively) inner sets and outer sets, which we may take to be increasing | decreasing (respectively), such that $\lim_n \mu G_n = \lim_n \mu F_n = \mu E$.

Proof Exercise.

Properties of Inner and Outer Sets and the Measure

3.2 Theorem

If (F_n) is a sequence of outer sets then $F = \bigcup_n F_n$ is an outer se

Moreover $\mu(\bigcup_n F_n) \leqslant \sum_n \mu F_n$.

Proof Each F_n is the union of countably many intervals and so F is a countabl

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Here's how

$$\begin{array}{llll}
I_{11} \rightarrow I_{12} & I_{13} \rightarrow I_{14} & \cdots \Rightarrow \\
I_{21} & I_{22} & I_{23} & \cdots \\
I_{31} & I_{32} & I_{33} & \cdots \\
I_{41} & \vdots & \cdots \\
\downarrow & & & \\
\vdots & & &
\end{array}$$

Now $F = \bigcup_{m,n} I_{m,m}$ and $I_{mm} \in \mathcal{B}$, so by Cor 2·6 $\mu F \leq \sum \mu I_{nm}$ where the sum is taken in the order indicated above. If $\sum_n \mu F_n = \infty$ then $\sum \mu I_{m,n}$ will diverge too and the result is true, trivially. If $\sum_n \mu F_n < \infty$ then the partial sums of $\sum \mu I_{nm}$ will be $\overline{\text{bd}}$. So $\sum \mu I_{mn}$ will converge. But so too does $\sum_n \sum_m \mu I_{mn}$ in this case and it is equal to $\sum \mu I_{nm}$ (**) hence,

$$\mu F \leq \sum \mu I_{nm} \leq \sum_n \sum_m \mu I_{nm} = \sum_n \mu F_n$$

$\square$

Any partial sum of $\sum \mu I_{nm}$ is $\leq \sum_{n=1}^{k} \left(\sum_m \mu I_{nm} \right)$ for some particular k

$$\leq \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \mu I_{nm} \right)$$

+ all we need is $\leq$!.

union of sets each of which is a countable union of intervals. Suppose $F_n = \bigcup_{m=1}^{\infty} I_{mn}$ with $I_{1n}, I_{2n}, \ldots$ a sequence of disjoint intervals for each $n \in \mathbb{N}$. The collection $\{I_{m,n} : m \in \mathbb{N}, n \in \mathbb{N}\}$ is <u>countable</u>, look

$$\begin{tikzcd}
I_{11} \arrow[r] & I_{21} \arrow[dl] & I_{31} \arrow[r] & I_{41} \arrow[dl] & \cdots \\
I_{12} \arrow[d] & I_{22} \arrow[ur] & I_{32} \arrow[dl] & I_{42} \arrow[ur] & \cdots \\
I_{13} \arrow[ur] & I_{23} \arrow[dl] & I_{33} \arrow[ur] & \cdots & \\
I_{14} \arrow[d] & I_{24} \arrow[ur] & \cdots & & \\
I_{15} \arrow[ur] & \cdots & & &
\end{tikzcd}$$

this way of counting produces a <u>sequence</u> of intervals which i'll call (G_n).

So F is the union of a sequence of intervals. So F is an outer set

If $\sum_n \mu F_n = \infty$ the other assertion of our Theorem is obviously true

If $\sum_n \mu F_n < \infty$ then as $\mu F_n = \sum_{m=1}^{\infty} \mu I_{mn}$ then

clearly $\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} \mu I_{mn}\right)$ converges because this is just $\sum_n \mu F_n$.

For the next step we need a theorem from second year analysis;

<u>Thm</u> (i) Let $\sum a_{ij}$ be a given double series and let (g_n) be an arrangement of the double sequence $\{a_{ij} ; i,j \in \mathbb{N}\}$ into a sequence. Then $\sum_{ij} a_{ij}$ converges absolutely $\Leftrightarrow \sum_n g_n$ converges absolutely. Moreover $\sum_i\left(\sum_j a_{ij}\right) = \sum_j\left(\sum_i a_{ij}\right) = \sum_{ij} a_{ij} = \sum_n g_n$.

(ii) Let (a_{ij}) be a double sequence. If for each fixed j, $\sum_{i=1}^{\infty} a_{ij}$ converges absolutely and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |a_{ij}|$ converges then, the double series $\sum_{ij} a_{ij}$ converges absolutely.

Ref Apostol 8·42 and 8·43 Mathematical Analysis 2nd Ed.

3.4 Theorem

(i) Let G, K be inner sets, then so is $K \cup G$. If $K \cap G = \emptyset$ then $\mu(G \cup K) = \mu G + \mu K$.

(ii) If F is an outer set and G an inner set then $F \setminus G$ is an outer set and $G \setminus F$ an inner set. If further $G \subseteq F$ then $\mu F = \mu G + \mu(F \setminus G)$ or if $F \subseteq G$ then $\mu G = \mu F + \mu G \setminus F$

Proof

(i) Let $E_n \in \mathcal{B}$ $E_n \uparrow \mathbb{R} \setminus G$ and $F_n \in \mathcal{B}$ $F_n \uparrow \mathbb{R} \setminus K$. Consider $E_n \cap F_n$, since each of (E_n) and (F_n) are increasing $E_n \cap F_n \uparrow$, $E_n \cap F_n \in \mathcal{B}$ and $\mathbb{R} \setminus G \cap \mathbb{R} \setminus K = \mathbb{R} \setminus G \cup K = \bigcup_n (E_n \cap F_n)$. So $\mathbb{R} \setminus G \cup K$ is an outer set. $G \cup K$ is clearly bounded & hence it is inner. Suppose I is an interval, $I \supseteq G \cup K$ and $G \cap K = \emptyset$. The measure of $G \cup K$ is given by $\lim_n \mu(I \setminus (E_n \cap F_n))$. Since

$\mu(I \setminus E_n \cap F_n) = \mu(I \setminus E_n \cup I \setminus F_n) = \mu(I \setminus E_n) + \mu(I \setminus F_n) - \mu[(I \setminus E_n) \cap (I \setminus F_n)]$

and $\lim_n \mu(I \setminus E_n) = \mu G$, $\lim_n \mu(I \setminus F_n) = \mu K$ it will be enough to show $\lim_n \mu[(I \setminus E_n) \cap (I \setminus F_n)] = 0$. Now $I \setminus E_n \cap I \setminus F_n = I \setminus E_n \cup F_n$ which decreases to $I \setminus \bigcup_n (E_n \cup F_n) = I \setminus (\mathbb{R} \setminus G \cup \mathbb{R} \setminus K) = I \setminus (\mathbb{R} \setminus G \cap K)$ but $G \cap K = \emptyset$ so $I \setminus \bigcup_n (E_n \cup F_n) = \emptyset$. The situation is this we have a sequence $I \setminus E_n \cap I \setminus F_n$ of basic sets decreasing to $\emptyset$. Now in 2.8 (ii) we showed that the value of $\lim_n \mu((I \setminus E_n) \cap (I \setminus F_n))$ depends only on what the sets decrease to and is equally well given by some other sequence of basic sets decreasing to $\emptyset$. Well $H_n = (-1/n, 0) \cup (0, 1/n)$ decreases to $\emptyset$ and $\lim_n \mu H_n = 0$:

(ii) If F is outer, G inner, $\exists$ $F_n \in \mathcal{B}$ $F_n \uparrow F$ and $G_n \in \mathcal{B}$ $G_n \downarrow G$

$F_n \setminus G_n \in \mathcal{B}$ $F_n \setminus G_n \uparrow F \setminus G$ so $F \setminus G$ is outer. $G_n \setminus F_n \in \mathcal{B}$ and $G_n \setminus F_n \downarrow G \setminus F$ which is clearly bounded and hence is inner, ~~If $G \subseteq F$ then as $F_n = G_n \cup F_n \setminus G_n$ then~~

M3P1 Correction 1

For 3·4 (ii) : If $G \subseteq F$ then $\mu F = \mu G + \mu(F \setminus G)$ or if

$F \subseteq G$ then $\mu G = \mu F + \mu(G \setminus F)$.

~~We've already shown $F \setminus G$ is outer~~, $G \setminus F$ is inner. [Stet ~~Suppose $G \subseteq F$~~.

Then $G \setminus F = \emptyset$ and if, as in first part of proof of 3·4 (ii), $F_n \uparrow F$, $G_n \downarrow G$ then $G_n \setminus F_n \downarrow G \setminus F = \emptyset$, further $\mu(G_n \setminus F_n) < \infty$ $\forall n$. So $\lim_n \mu(G_n \setminus F_n) = 0$. Now

$$G_n = G_n \setminus F_n \cup (F_n \cap G_n)$$

So $\mu G_n = \mu(G_n \setminus F_n) + \mu(F_n \cap G_n)$

thus $\lim_n \mu(F_n \cap G_n)$ exists and is equal to $\lim_n \mu G_n = \mu G$.

Now $F_n = F_n \setminus G_n \cup (F_n \cap G_n)$ and $F_n \setminus G_n \uparrow F \setminus G$

So
$$\begin{array}{ccccc} \mu F_n & = & \mu(F_n \setminus G_n) & + & \mu(F_n \cap G_n) \\ \downarrow & & \downarrow & & \downarrow \\ \mu F & = & \mu(F \setminus G) & + & \mu(G) \end{array}$$

Exercise: prove the corresponding result for $F \subseteq G$.

3.5 Theorem

A set E is integrable if and only if $\forall \epsilon > 0$ $\exists$ F outer and G inner such that $F \supseteq E \supseteq G$ and $\mu(F \setminus G) < \epsilon$.

Proof If E is integrable $\inf_{\substack{F \text{ outer} \\ F \supseteq E}} \mu F = \sup_{\substack{G \text{ inner} \\ G \subseteq E}} \mu G$ and an $\frac{\epsilon}{2}$ argument will get the result. Conversely we have by 3.4 that $\mu(F \setminus G) = \mu F - \mu G$ and hence $\inf_{\substack{F \text{ outer} \\ F \supseteq E}} \mu F - \sup_{\substack{G \text{ inner} \\ G \subseteq E}} \mu G \leqslant \mu F - \mu G < \epsilon$. Since ϵ is arbitrary $\mu_* E = \mu^* E$ & E is integrable. $\square$

$$\begin{array}{ccccc} \mu F_n - \mu G_n & = & \mu F_n \setminus G_n & & \\ \downarrow \quad\quad \downarrow & & \downarrow & & \\ \mu F - \mu G & = & \mu F \setminus G & & \text{as } n \to \infty . \end{array}$$

The other part is similar. $\square$.

3·5 Characterisation of Integrable sets. see pp $20\frac{1}{2}$

Properties of Integrable Sets

3·6 Theorem

(i) If (E_n) is a sequence of disjoint integrable sets and $\sum_n \mu E_n < \infty$ then $E = \bigcup_n E_n$ is integrable and $\mu E = \sum_n \mu E_n$.

(ii) If E and F are integrable then so is $E \setminus F$.

(iii) If E and F are integrable then so is $E \cap F$.

Proof

Let $\epsilon > 0$. Choose outer sets $F_n \supseteq E_n$ with $\mu F_n < \mu E_n + \epsilon/2^n$ then $F = \bigcup_n F_n$ is an outer set $F \supseteq E$ and $\mu F \leqslant \sum_n \mu F_n < \sum_n \mu E_n + \epsilon$.

Hence $\mu^* E \leqslant \sum_n \mu E_n$.

Choose m so that $\sum_{n=1}^m \mu E_n \geqslant \sum_n \mu E_n - \epsilon/2$ and inner sets $K_n \subseteq E_n$ with $\mu K_n \geqslant \mu E_n - \epsilon/2m$ for $n = 1, 2, \ldots, m$. Then by 3·4 $K = \bigcup_{n=1}^m K_n$ is inner and $K \subseteq \bigcup_{n=1}^m E_n$. Now $\mu K = \sum_{n=1}^m \mu K_n$ because the K_n's are disjoint (this follows because the E_n's are disjoint). So $\mu K > \sum_{n=1}^m (\mu E_n - \epsilon/2m) = \sum_{n=1}^m \mu E_n - \epsilon/2 \geqslant \sum_{n=1}^\infty \mu E_n - \epsilon$. Since K is inner this means that $\mu_* E \geqslant \sum_n \mu E_n$. Since we know that $\mu_* E \leqslant \mu^* E$ <u>always</u>, we have $\mu_* E = \mu^* E = \sum_n \mu E_n$ i.e. E is integrable and $\mu E = \sum_n \mu E_n$.

(ii) Let $\epsilon > 0$, Choose G outer and H inner with $G \supseteq E \supseteq H$
$\mu G < \mu E + \epsilon/4$, $\mu H > \mu E - \epsilon/4$. And similarly $J \supseteq F \supseteq K$

J outer, K inner and $\mu J < \mu H + \epsilon/4$, $\mu K > \mu G - \epsilon/4$.

Now (check this)

$$G \setminus K \supseteq E \setminus F \supseteq H \setminus J \qquad \text{(think in terms of sizes)}$$

and by 3·4 $G \setminus K$ is outer, $H \setminus J$ is inner. Now

$$\begin{aligned}
(G \setminus K) \setminus (H \setminus J) &= (G \cap K') \setminus (H \cap J') \\
&= (G \cap K') \cap (H \cap J')' \\
&= (G \cap K') \cap (H' \cup J) \\
&= G \cap K' \cap H' \cup G \cap K' \cap J \\
&\subseteq G \cap H' \cup J \cap K' \\
&= G \setminus H \cup J \setminus K
\end{aligned}$$

So $\mu((G \setminus K) \setminus (H \setminus J)) \leqslant \mu(G \setminus H) + \mu(J \setminus K)$

but by 3·4 (ii) this is $= (\mu G - \mu H) + (\mu J - \mu K)$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\square$

(iii) $E \cap F = E \setminus (E \setminus F)$

J outer, K inner and $\mu J < \mu F + \epsilon$, $\mu K > \mu F - \epsilon$.

Now $G \setminus K \supseteq E \setminus F \supseteq H \setminus J$ (think in terms of sizes). By 3·4 $G\setminus K$ is outer and $H \setminus J$ is inner. Also $\mu(G\setminus K) = \mu G - \mu K \overset{\text{NO}}{<}$ $< \mu E + \epsilon - (\mu F - \epsilon) = \mu E - \mu F + 2\epsilon$ and $\mu(H\setminus J) = \mu H - \mu J >$ $(\mu E - \epsilon) - (\mu F + \epsilon) = \mu E - \mu F - 2\epsilon$. Hence $(G\setminus K)\setminus(H\setminus J)$ is an outer set and $\mu((G\setminus K)\setminus(H\setminus J)) = \mu(G\setminus K) - \mu(H \setminus J) < (\mu E - \mu F) + 2\epsilon - (\mu E - \mu F - 2\epsilon)$ $= 4\epsilon$. Adjusting ϵ gives the result via 3·5.

$\square$

(iii) $E \cap F = E \setminus (E \setminus F)$

$$(G\setminus K)\setminus(H\setminus J) = (G\cap K')\setminus(H\cap J') = (G\cap K')\cap(H\cap J')'$$
$$= (G\cap K')\cap(H\cup J) \subseteq \underbrace{(G\cap K'\cap H')}_{G\setminus H} \cup \underbrace{(G\cap K'\cap}_{J\setminus K}$$

Summary

An outer set is the union of a sequence of basic sets. One can unambiguously define the measure of an outer set E via the measure of a (any!) sequence of basic sets which increase to E. Any open set is outer. An inner set is a bounded set whose complement is outer. One can unambiguously define the measure of an inner set. A set is integrable if it sits between an outer (enveloping) set and an inner (subset) whose measures can be made as close as we like.

The class of outer sets is closed under countable unions. The measure is subadditive on countable unions. If μ acts on a countable disjoint union then it is countabley additive. μ is additive on finite disjoint unions of inner sets.

Integrable sets are closed under countable unions for which "the series of measures converges", differences and intersections. The measure is countably additive on disjoint unions (so long as the series of measures converges)