4 The Lebesgue Integral

4.0 The pattern of development for the integral is "very" similar to that for the measure. We shall define the integral for "simple functions" we shall extend it to a more complicated class — the upper (read outer) functions. We will define the dual idea of lower (read inner) functions and use these two types of functions to identify Integrable functions.

4.1 Step Functions; Some Definitions

(i) Let $E \subseteq \mathbb{R}$, the characteristic function of E, or indicator function of E is simply the function

$$I_E(s) = \chi_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E \end{cases}.$$

(ii) A step function is a linear combination of characteristic functions. Equivalently; f is a step function if $\exists\ c_1, \dots, c_n \in \mathbb{R}$ and disjoint intervals $I_1, \dots, I_n$ such that $f(s) = \begin{cases} c_j & \text{if } s \in I_j \\ 0 & \text{if } s \in \mathbb{R} \setminus \bigcup_{j=1}^{n} I_j \end{cases}$.

We can write $f(s) = \sum_{j=1}^{n} c_j \chi_{I_j}(s)$. We note that there is more than one way of writing any given f as a linear combination of characteristic functions. We let the set of all step functions be denoted by S.

4.2 Proposition

(i) Under pointwise sums and with scalar multiplication S become a linear space.

(ii) Let $\mathcal{B}$ denote basic sets as in §1. Then $\chi_E \in S \iff E \in \mathcal{B}$.

(iii) With $\mathcal{B}$ as in (ii), If $f \in S$ then $\{f > a\} \in \mathcal{B}$ for each $a \in \mathbb{R}$

(iv) If $f \in S$ & $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$, then $f^+, f^-, |f| = f^+ + f^-, \in S$

(v) If $f \in S$, $E \in \mathcal{B}$ then $f\chi_E \in S$.

Proof Exercise

4.3 Definition

Let $f \in S$ and suppose $f(s) = \sum_{j=1}^{n} c_j \chi_{I_j}(s)$ $I_r \cap I_t = \emptyset$ if $r \neq t$. Then

$$\int_{\mathbb{R}} f \, d\mu \overset{\text{def}}{=} \sum_{j=1}^{n} c_j \mu I_j .$$

The following result shows that $\int_{\mathbb{R}} f \, d\mu$ is independant of how we write f.

4.4 Proposition (The Integral is well defined).

Suppose $f \in S$ and $f(s) = \sum_{j=1}^{n} c_j \chi_{I_j}(s) = \sum_{\ell=1}^{m} d_\ell \chi_{K_\ell}(s)$.

Then $\sum_{j=1}^{n} c_j \mu I_j = \sum_{\ell=1}^{m} d_\ell \mu K_\ell$.

Proof

We suppose that $\{I_j\}$ and $\{K_\ell\}$ are families of disjoint intervals.

Let $H_{j\ell} = I_j \cap K_\ell$ then $\{H_{j\ell} : 1 \leq j \leq n, 1 \leq \ell \leq m\}$ is a family of disjoint intervals. We ~~have~~ can assume that $\bigcup_j I_j = \bigcup_\ell K_\ell$ (if not just add a few intervals to the I_j's + K_ℓ's) and hence

$\bigcup_j H_{j\ell} = (\bigcup_j I_j) \cap K_\ell = K_\ell$ and $\bigcup_\ell H_{j\ell} = I_j \cap (\bigcup_\ell K_\ell) = I_j$.

So $f(s) = \sum_{j=1}^{n} c_j \left(\sum_{\ell=1}^{m} \chi_{H_{j\ell}}(s)\right) = \sum_{j=1}^{n}\sum_{\ell=1}^{m} c_j \chi_{H_{j\ell}}(s) = \sum_{\ell=1}^{m}\sum_{j=1}^{n} d_\ell \chi_{H_{j\ell}}(s)$

This shows that, for example, $f(s) = c_1$ on $H_{11} \cup H_{12} \cup \dots \cup H_{1m}$ and $f(s) = d_1$ on $H_{11} \cup H_{21} \cup \dots \cup H_{n1}$ and hence $c_1 = d_1$ (must agree on H_{11}!) or $H_{11} = \emptyset$ by similar arguments one shows that either the number multiplying $\chi_{H_{j\ell}}$ in each of the double sums above are equal or $\mu H_{j\ell} = 0$. It now follows, by additivity of μ that

$$\sum_j c_j \mu I_j = \sum_\ell d_\ell \mu K_\ell . \qquad \square$$

4.5 Proposition Properties of the integral on S

(i) If $f \geq 0$, $f \in S$ then $\int_{\mathbb{R}} f \geq 0$.

(ii) If $f, g \in S$ then $\int_{\mathbb{R}} f+g = \int_{\mathbb{R}} f + \int_{\mathbb{R}} g$.

(iii) If $f \in S$, $c \in \mathbb{R}$ then $\int_{\mathbb{R}} cf = c\int_{\mathbb{R}} f$.

(iv) If $f \leqslant g$, $f, g \in S$ then $\int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} g$.

(v) $\left|\int_{\mathbb{R}} f\right| \leqslant \int_{\mathbb{R}} |f|$.

(vi) $\exists\ f \in S$, $f \not\equiv 0$, s.t. $f \geqslant 0$ but $\int_{\mathbb{R}} f = 0$. Hence $f \longmapsto \int |f|$ is <u>not</u> a norm (it <u>is</u> a semi-norm).

<u>Proof</u> (i) exercise

(ii) Let $f = \sum_j c_j \chi_{I_j}$ $\quad g = \sum_\ell d_\ell \chi_{K_\ell}$ $\quad H_{j\ell} = I_j \cap K_\ell$ (disjoint family)

and assume $\bigcup_j I_j = \bigcup_\ell K_\ell$ (w.l.o.g.).

Then $f+g = \sum_j\sum_\ell (c_j + d_\ell)\chi_{H_{j\ell}}$ and $\int_{\mathbb{R}} f+g = \sum_j\sum_\ell (c_j+d_\ell)\mu H_{j\ell}$

$$= \sum_j\sum_\ell c_j \mu H_{j\ell} + \sum_j\sum_\ell d_\ell \mu H_{j\ell}$$

$$= \int_{\mathbb{R}} f + \int_{\mathbb{R}} g$$

(iii) Obvious

(iv) exercise $\quad$ ($f \leqslant g \iff \forall s \in \mathbb{R}\ \ f(s) \leqslant g(s)$)

(v) $|f|$ is in S and so is $-|f|$ and $-|f| \leqslant f \leqslant |f|$

So $-\int_{\mathbb{R}} |f| \leqslant \int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} |f|$ by (iv) so $\int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} |f|$ <u>and</u> $-\int_{\mathbb{R}} f \leqslant \int |f|$

hence $\left|\int_{\mathbb{R}} f\right| \leqslant \int |f|$.

(vi) Exercise.

4·6 Remark

Our goal is to define an integral for a wide class of function. Our strategy is to define the integral for simple functions, S, and then extend it, gradually, through wider and wider classes of functions. We are going to look first at functions which are "limits" of increasing sequences of step functions. We will need some other results for the development of the integral and they are included here too.

4·7 Lemma

If (f_n) is a decreasing sequence of step functions and $\forall x \quad f_n(x) \downarrow 0$ as $n \to \infty$ then $\int_{\mathbb{R}} f_n \to 0$.

Proof

$\forall n \quad f_n \leqslant f_1$ and f_1 is bounded and $\{f_1 \neq 0\} \in \mathcal{B}$. Thus $\exists \lambda > 0$ and an interval I : $\forall n \forall x, \lambda \geqslant f_n(x)$, $f_n(x) = 0 \quad x \in \mathbb{R}\setminus I$. Let $\epsilon > 0$ and $E_n = \{f_n \geqslant \epsilon\}$. Then $E_n \in \mathcal{B}$ (4·2(iii)) and $\bigcap_n E_n = \{x : \forall n \ f_n(x) \geqslant \epsilon\}$ $= \{x : \lim_n f_n(x) \geqslant \epsilon > 0\} = \emptyset$ by hypothesis. Now $f_{n+1} \leqslant f_n$ and so $E_{n+1} \subseteq E_n$. So (E_n) is a decreasing sequence of basic sets with limit $\emptyset$ so $\lim_n \mu E_n = 0$ (see Proof, (i), 3·4). Now

$$\int_{\mathbb{R}} f_n = \int_{\mathbb{R}\setminus I} f_n + \int_{E_n} f_n + \int_{I\setminus E_n} f_n \qquad \text{(see Exercises)}$$

$$\leqq 0 + \lambda \mu E_n + \epsilon \mu(I\setminus E_n)$$

So $\lim_n \int_{\mathbb{R}} f_n \leqslant \epsilon \mu(I)$. And $\epsilon > 0$ is arbitrary. So the left hand side is 0.

4.8 Proposition

Let (f_n) and (g_n) be increasing sequences of step functions with $\lim_n f_n(x) = \lim_n g_n(x) \leqslant \infty$. Then $\lim_n \int g_n = \lim_n \int f_n$.

Proof

Let $m \in \mathbb{N}$ be fixed and consider $(f_n - g_m)$. Since $f_n \uparrow$ then $(f_n - g_m) \uparrow$. moreover $\forall x$

$$\lim_n \left(f_n(x) - g_m(x) \right) = \left(\lim_n f_n(x)\right) - g_m(x) = \left(\lim_n g_n(x)\right) - g_m(x)$$

(by hypothesis). But (g_n) is an <u>increasing</u> sequence, so $\forall x$

$$\lim_n \left(f_n(x) - g_m(x)\right) = \left(\lim_n g_n(x)\right) - g_m(x) \geqslant 0 .$$

Now consider the positive and negative parts of $(f_n - g_m)$. Since $f_n - g_m$ is increasing it follows that $(f_n - g_m)^+ \uparrow$ and $(f_n - g_m)^- \downarrow$. Now $\forall x$ $\lim_n \left(f_n(x) - g_m(x)\right) \geqslant 0$ so if y is such that $\lim_n (f_n - g_m)^-(y) > 0$ then $\lim_n \left(f_n(y) - g_m(y)\right) < 0$, a contradiction, so we have $\lim_n (f_n - g_m)^-(x) = 0$ $\forall x$. We can use 4·7 to conclude that $\lim_n \int_{\mathbb{R}} (f_n - g_m)^- = 0$.

Now $(f_n - g_m) \uparrow$ so therefore does $\int_{\mathbb{R}} (f_n - g_m)$ and since

$$f_n - g_m = (f_n - g_m)^+ - (f_n - g_m)^- \quad \text{then}$$

$$\lim_n \int_{\mathbb{R}} (f_n - g_m) = \lim_n \int_{\mathbb{R}} (f_n - g_m)^+ - \lim_n \int_{\mathbb{R}} (f_n - g_m)^- = \lim_n \int_{\mathbb{R}} (f_n - g_m)^+ \geqslant 0$$

Hence $\lim_n \int_{\mathbb{R}} f_n \geqslant \int_{\mathbb{R}} g_m$, but the g_m's increase so,

$$\lim_n \int_{\mathbb{R}} f_n \geqslant \lim_m \int_{\mathbb{R}} g_m .$$

Note that this argument is "symmetric in f and g" to conclude $\lim_m \int_{\mathbb{R}} g_m \geq \lim_n \int_{\mathbb{R}} f_n$.

$\square$

4·9 Definition

A function $f : \mathbb{R} \to \mathbb{R}^*$ is an <u>upper function</u> if and only if $\exists$ a sequence of step functions (f_n) with $f_n \uparrow f$. An upper function f is <u>integrable</u> if $\sup_n \int_{\mathbb{R}} f_n < \infty$. In this case we define $\int_{\mathbb{R}} f = \sup_n \int_{\mathbb{R}} f_n$.

4·10 Corollary

If f is an integrable upper function then $f \mapsto \int_{\mathbb{R}} f$ is a well defined function.

<u>Proof</u> By 4·8 $\square$.

4·11 Theorem

Let f be an integrable upper function. Then $\mu\{x : f(x) = \infty\} = 0$. That is, f is finite almost every-where.

Let $f_n \uparrow f$ as in 4·9.

<u>Proof</u> Assume first that $\forall n f_n \geq 0$. Let $E_{n,m} = \{x : f_n(x) > m\}$ by 4·2 (iii) $E_{n,m}$ is a basic set. Now let $E_m = \bigcup_n E_{n,m}$ $= \{x : \exists n \in \mathbb{N} : f_n(x) > m\}$. Since $f_n \uparrow$ we have $E_m = \{x : \lim_n f(x) = f(x) > m\}$ as it is clearly an outer set. Now let $E = \bigcap_m E_m = \{x : f(x) > m \ \forall m \in \mathbb{N}\} = \{f = \infty\}$.

Our goal must be to show that E is integrable and $\mu E = 0$.

Let $M = \sup_n \int_{\mathbb{R}} f_n < \infty$ (by hypothesis). For the moment fix $n \in \mathbb{N}$ and consider $f_n \restriction E_{n,m}$, <u>which is a step function</u>, as $E_{nm} = \{ f_n > m \}$ then $f_n \chi_{E_{n,m}} \geq m \chi_{E_{n,m}}$ $\longleftarrow$ this is a step funct

too! . Hence $\int_{\mathbb{R}} f_n \chi_{E_{nm}} = \int_{E_{n,m}} f_n \geq m \int_{\mathbb{R}} \chi_{E_{n,m}} = m \mu E_{n,m}$.

So that $M \geq m \mu E_{n,m}$. Now $f_n \uparrow$ and so $(E_{n,m})$ increas
(with n for a fixed m) . Hence as $E_m = \bigcup_n E_{n,m}$ and E_m is
outer then $\mu E_m = \lim_n \mu E_{n,m} \leq \dfrac{M}{m}$. So we have the
following; $\phi \subseteq E \subseteq E_m$ and $\mu E_m \leq \dfrac{M}{m} \longrightarrow 0$ as $m \to \infty$

So E is $\overset{\text{inner}}{\text{integrable}}$ $\overset{\text{outer}}{\text{and}}$ $\mu E = 0$.

Suppose now that (f_n) are not necessarily all positive. Then
consider $g_n = f_n - f_1 \geq 0$. We have $g_n \geq 0$, $g_n \uparrow f - f_1$
and $\sup_n \int_{\mathbb{R}} g_n = \left(\sup_n \int_{\mathbb{R}} f_n \right) - \int_{\mathbb{R}} f_1$. So by the first part
$\mu \{ x : f(x) - f_1(x) = \infty \} = 0$. Thats enough. $\square$.

4:12 "Corollary"

If $f_n \uparrow$ and $\sup_n \int_{\mathbb{R}} f_n < \infty$ then by defining $g_1 = f_1, \ldots$
$\ldots g_{n+1} = f_{n+1} - f_n \geq 0$ then $f_n = (f_n - f_{n-1}) + (f_{n-1} - f_{n-2}) + \cdots (f_2 - f_1) + f_1$
$= \sum_{k=1}^{n} g_k$ and $\int_{\mathbb{R}} f_n = \sum_{k=1}^{n} \int_{\mathbb{R}} g_k$. So we may restate 4.11
as follows; Suppose (g_k) is a sequence of step functions with
$\sum_k \int_{\mathbb{R}} g_k < \infty$ and $g_k \geq 0$ $\forall k$. Then $g(x) = \sum_{k=1}^{\infty} g_k(x)$ is
finite "a.e.", and
an integrable upper function, $g(x)$ is
$\int_{\mathbb{R}} g = \sum_k \int_{\mathbb{R}} g_k$. $\square$

We are now going to define the analogue of an inner set, for convenience we will define it via upper functions just as we defined inner sets via outer sets.

4:13 Definitions

(i) $f : \mathbb{R} \to \mathbb{R}^*$ is a <u>lower function</u> if and only if $-f$ is an upper function. A lower function f is <u>integrable</u> if $-f$ is an integrable upper function and we define

$$\int_{\mathbb{R}} f \overset{\text{def}}{=} -\int_{\mathbb{R}} -f \quad \text{(this defined as in 4.9)}.$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}^*$ we define the <u>upper integral</u> of f by

$$\int_{\mathbb{R}}^{*} f = \inf \left\{ \int_{\mathbb{R}} g \; : \; g \geq f \text{ and } g \text{ is upper} \right\}$$

(iii) Let $f : \mathbb{R} \to \mathbb{R}^*$ we define the <u>lower integral</u> of f

$$\int_{*\mathbb{R}} f = \operatorname{Sup} \left\{ \int_{\mathbb{R}} h \; : \; h \leq f, \; h \text{ a lower function} \right\}$$

(iv) Let $f : \mathbb{R} \to \mathbb{R}^*$ then f is <u>integrable</u> if and only if

$$\infty > \int_{\mathbb{R}}^{*} f = \int_{*\mathbb{R}} f.$$

In this case $\int_{\mathbb{R}} f \overset{\text{def}}{=} \int_{*\mathbb{R}} f = \int_{\mathbb{R}}^{*} f$.

4:14 Remarks

(a) We denote the class of integrable functions by $\mathcal{L}$, if we wish to emphasise that they are functions $\mathbb{R} \to \mathbb{R}^*$ then we write $\mathcal{L}(\mathbb{R})$.

M3P1 - (JANE THIS IS A HAND OUT NOT PART OF THE TEXT)

The extended real numbers, $\overline{\mathbb{R}}^*$, are just $\mathbb{R} \cup \{-\infty, +\infty\}$. The algebraic operations are given below. In your notes it may happen that I've omitted to consider the possibility that (say) $\mu E = \infty$ for some E. In this event the case $\mu E = \infty$ has to be considered separately and you should do this yourself. I should add that I <u>have</u> made a point of considering the cases where some quantity is infinite so the number of omissions ought to be small. Let $x \in \mathbb{R}$

(i) $(\pm\infty) + (\pm\infty) = x + (\pm\infty) = (\pm\infty) + x = \pm\infty$

(ii)
$$x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \mp\infty & \text{if } x < 0 \end{cases}$$

(iii) $(\pm\infty)(\pm\infty) = +\infty$

(iv) $(\pm\infty)(\mp\infty) = -\infty$

(v) $x/(\pm\infty) = 0$

(vi) $-\infty < x < +\infty$

AND $\infty - \infty$ is meaningless!

(β) One can show that $\displaystyle\int_{*\mathbb{R}} f = -\int_{\mathbb{R}}^{*} -f$.

In order to develop the theory of integrable functions we need some technical results on upper / lower functions. We will finish this section with these. You will <u>not</u> be examined on the <u>proofs</u> of 4:16 and 4:17 but you <u>must</u> understand the results

<u>4:15 Lemma</u> (Revise this)

Suppose h is lower, g is upper and $h \leqslant g$ then $\displaystyle\int_{\mathbb{R}} h \leqslant \int_{\mathbb{R}} g$ so long as one of $\int_{\mathbb{R}} h, \int_{\mathbb{R}} g$ is finite.

<u>Proof</u> The proof of this result is like that of 3.4 (ii).

Consider $g-h$. Since h is lower, $-h$ is upper and $\exists\, h_n \in S : h_n \uparrow -h$ thus $-h_n \downarrow h$, i.e. h is the pointwise limit of a decreasing sequence of step functions. Suppose that $g_n \uparrow g$ and $k_n \downarrow h$, $g_n, k_n \in S$. Then $g_n - k_n \uparrow g-h$ so $g-h$ is an upper function. Suppose now that we know the following result; If ℓ is an upper function and $\ell \geqslant 0$ then $\int_{\mathbb{R}} \ell \geqslant 0$. Then we could conclude that $\int_{\mathbb{R}} g-h \geqslant 0$. Now

$$0 \leqslant \int_{\mathbb{R}} g-h = \lim_n \int_{\mathbb{R}} g_n - k_n = \lim_n \int_{\mathbb{R}} g_n - \lim_n \int_{\mathbb{R}} k_n$$

$$= \lim_n \int_{\mathbb{R}} g_n - \lim_n \left(-\int_{\mathbb{R}} -k_n\right)$$

$$= \int_{\mathbb{R}} g - \int_{\mathbb{R}} h$$

So that $\int_{\mathbb{R}} g \geqslant \int_{\mathbb{R}} h$. So lets prove the result for ℓ described above. Suppose $\ell_n \in S$ and $\ell_n \uparrow \ell \geqslant 0$, now (as in 4.8 !) the map $x \longmapsto x^{\pm}$ is cts so $\ell_n^+ \uparrow \ell^+ = \ell$ (because $\ell \geqslant 0$)

Since $\ell_n \uparrow$ $\ell_n^- \downarrow$ and $\ell_n^- \downarrow \ell^- = 0$, so $\int_{\mathbb{R}} \ell = \lim_n \int_{\mathbb{R}} \ell_n$ =

$$\lim_n \int_{\mathbb{R}} \ell_n^+ - \lim_n \int_{\mathbb{R}} \ell_n^- = \lim_n \int_{\mathbb{R}} \ell_n^+ \geqslant 0. \quad \square.$$

4:16 Lemma

If (f_n) is an increasing sequence of upper functions then $f = \lim_n f_n$ is an upper function and $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

Proof k_n ?

Let $f_{nk} \uparrow f_n$ as $k \to \infty$. STET Let $g_n = \max\{f_{in} : 1 \leqslant i \leqslant n\}$ max{first n of column n}

ie. $g_1 = \max\{f_{11}\}$, $g_2 = \max\{f_{12}, f_{22}\}$ $g_3 = \max\{f_{13}, f_{23}, f_{33}\}$

etc, by considering the array

$$\begin{array}{ccccccccc}
f_{11} & \leqslant & f_{12} & \leqslant & f_{13} & \leqslant & \cdots & \leqslant & f_1 \\
 & & & & & & & & \wedge| \\
f_{21} & \leqslant & f_{22} & \leqslant & f_{23} & \leqslant & \cdots & \leqslant & f_2 \\
 & & & & & & & & \wedge| \\
f_{31} & \leqslant & f_{32} & \leqslant & f_{33} & \leqslant & \cdots & \leqslant & f_3 \\
\vdots & & \vdots & & \vdots & & & &
\end{array}$$

it is clear that (i) $g_1 \leqslant g_2 \leqslant \cdots g_n \leqslant g_{n+1} \leqslant \cdots$

(ii) For a fixed n and any $k \geqslant n$ $g_k \geqslant f_{nk}$

Hence $\lim_k g_k \geqslant \lim_k f_{nk} = f_n$ $\forall n$, hence $\lim_k g_k \geqslant \lim_n f_n = f$

On the other hand $g_n = \max_{1 \le i \le n} f_{in} \leqslant \max_{1 \le i \le n} f_i = f_n$ so that $\lim_n g_n \leqslant \lim_n f_n = f$. So f is an upper function. By definition $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} g_n$. But $\forall n$ $g_n \leqslant f_n$ so $\lim_n \int_{\mathbb{R}} g_n \leqslant \lim_n \int_{\mathbb{R}} f_n$

Also $f_{nk} \leqslant g_\ell$ where $\ell = \max\{n,k\}$ so $\int_{\mathbb{R}} f_{nk} \leqslant \int_{\mathbb{R}} g_\ell$ and $\lim_k \int_{\mathbb{R}} f_{nk} \leqslant \lim_k \int_{\mathbb{R}} g_\ell$;
(note that ℓ is a function of k and $\ell \to \infty$ as $k \to \infty$).
But $\lim_k \int_{\mathbb{R}} f_{nk} = \int_{\mathbb{R}} f_n$ (by definition) so $\lim_k \int_{\mathbb{R}} f_k \leqslant \lim_k \int_{\mathbb{R}} g_\ell$
$= \lim_\ell \int_{\mathbb{R}} g_\ell$. So $\lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$. $\square$.

4:17 Lemma

Let $N \subseteq \mathbb{R}$ and suppose $\mu N = 0$. Then $\forall \epsilon > 0$ $\exists$ an upper function f with,

(i) $f \geqslant 0$.

(ii) $f(x) = \infty$ for $x \in N$.

(iii) $\int_{\mathbb{R}} f < \epsilon$.

Proof

By assumption N is finitely measurable, thus $\forall n \in \mathbb{N}$ $\exists$ an outer set $G_n \supseteq N$: $\mu G_n < 1/2^{n+1}$. We may consider each G_n as the union of a sequence of disjoint intervals (I_{kn}) and thus we may consider χ_{G_n} as the limit of the increasing sequence of step functions $\sum_{k=1}^{m} \chi_{I_{kn}}$. So χ_{G_n} is an upper function and (by 4:16 !) $\int_{\mathbb{R}} \chi_{G_n} < 1/2^n$. Let $f_k = \sum_{n=k}^{\infty} \chi_{G_n} = \lim_m \sum_{n=k}^{m} \chi_{G_n}$.

Then $\left(\sum_{n=k}^{m} \chi_{G_n}\right)$ is an increasing (with m) sequence of upper functions with limit f_k, thus by 4:16 f_k is an upper function and $\int_{\mathbb{R}} f_k = \lim_m \int_{\mathbb{R}} \left(\sum_{n=k}^{m} \chi_{G_n} \right) = \lim_m \sum_{n=k}^{m} \mu G_n < \lim_m \sum_{n=k}^{m} \frac{1}{2^{n+1}} < \frac{1}{2^k}$

Clearly then f_k satisfys (i) $\mathrm{dom}(f_k) \supseteq N$

(ii) $f_k \geqslant 0$

(iii) $\int_{\mathbb{R}} f_k < 1/2^k$

and finally if $x \in N$, $f_k(x) = \sum_{k=n}^{\infty} \chi_{G_n}(x) = \sum_{n=k}^{\infty} 1 = \infty$

□

4:18 Lemma

Suppose E is an outer set with $\mu E < \infty$. Then χ_E is an integrable upper function. In fact, more generally, we have $f \geqslant 0$ is an integrable upper function and E an outer set of finite measure then $f \cdot \chi_E$ is an integrable upper function. Moreover if $f \geqslant 0$

$$\int_{\mathbb{R}} f \geqslant \int_E f \stackrel{\text{def}}{=\!=} \int_{\mathbb{R}} f \cdot \chi_E .$$

take this out it's unnecessary (see 5·6)

Proof (Suppose first that $\forall n$ $f_n \geqslant 0$) Let $f_n \in S$, $f_n \uparrow f$ and $E_n \in \mathcal{B}$ $E_n \uparrow E$. Then $g_n = f_n \cdot \chi_{E_n} \in S$, $g_n \uparrow f \cdot \chi_E$ i.e. $f\chi_E$ is outer. Clearly $g_n \leqslant f_n$,

So $\int_{\mathbb{R}} g_n \leqslant \int_{\mathbb{R}} f_n$ + hence $\lim_n \int_{\mathbb{R}} g_n = \int_{\mathbb{R}} f \cdot \chi_E \leqslant \lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$

If f_n's are not +ve then take (f_n^+), these are +ve and increase to f. The first statement follows by considering E as a disjoint union of intervals (I_n), then $\chi_E = \lim_m \sum_{n=1}^{m} \chi_{I_n}$ and $\left(\sum_{n=1}^{m} \chi_{I_n}\right)$ is increasing and $\int_{\mathbb{R}} \sum_{n=1}^{m} \chi_{I_n} = \sum_{n=1}^{m} \mu I_n \uparrow \mu E$.

~~Remark The condition $f \geqslant 0$ is unnecessary, consider $f^+ + f$~~