# MRP – Proof

Using all of our results on orthogonal martingales we can consider the stable subspace of $\mathcal{M}^2_0(T)$ generated by $(W_t)$. As we have seen this is the closed subspace given by all stochastic integrals (of suitable integrands) with respect to $W$. Let $Y$ be a martingale in the stable subspace orthogonal to that generated by $W$.

Let $\sigma_n = \inf\{t : |Y_t| \geq n\}$ and set $Y^n_t = \frac{1}{2n} Y_{\sigma_n \wedge t}$. Then for each $n$, $(Y^n_t)$ is in the stable subspace strongly orthogonal to that generated by $W$, see Theorem . So, $|Y^n| \leq \frac{1}{2}$ and $(Y_t)$ is orthogonal to both $(W_t)_t$ and $(W_t^2 - t) \equiv (2\int_0^t W dW)$. Recalling that $Y_0 = 0$, and therefore $Y^n_0 = 0$ too, we define

$$Q(E) = \int_E (1 + Y^n_T)\, d\mathbb{P}$$

Note that since $|Y^n| \leq \frac{1}{2}$ then $1 + Y^n_T$ is a strictly positive random variable and $Q$ is certainly a measure (Radon-Nikodym Theorem). Moreover

$$Q(\Omega) = \int_\Omega (1 + Y^n_T)\, d\mathbb{P}$$

$$= 1 + \int_\Omega Y^n_T \, d\mathbb{P}$$

$$= 1 + \int_\Omega M_0^{\mathbb{P}}(Y_T^n)\, d\mathbb{P}$$

$$= 1 + 0$$

So $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$.
Consider, first of all, the product

$$W_t(1 + Y_t^n) = W_t + W_t \cdot Y_t^n$$

this is the sum of two $\mathbb{P}$-martingales, $(W_t)$ and (since $Y$ is strongly orthogonal to $W$) $W_t \cdot Y_t^n$. Recall our characterisation of '$\mathbb{Q}$' martingales in the section on changes of measure. This is telling us that $(W_t)$ is a (continuous) $\mathbb{Q}$ martingale. A similar argument shows that $W_t^2 - t$ is a continuous $\mathbb{Q}$ martingale. Recall now our Corollary to Levy's characterisation: this is telling us that $W$ is a $\mathbb{Q}$ Brownian Motion!!!. So, the $\mathbb{Q}$ distribution of $W_t$ must agree with the $\mathbb{P}$ distribution of $W_t$, i.e.

$$\mathbb{P}\{W_t^{-1}(H)\} = \mathbb{Q}\{W_t^{-1}(H)\}$$

for every $t \in [0, T]$ and Borel set $H \subseteq \mathbb{R}$. So $\mathbb{P}$ and $\mathbb{Q}$ agree on the generating sets for $\mathcal{F}_T$. It is a short argument to show they agree on $\mathcal{F}_T$ (*) (H.C.-Extension). For any $\mathcal{F}_T$ set, $E$, then,

$$\mathbb{P}(E) = \mathbb{Q}(E) = \mathbb{P}(E) + \int_E Y_T^n\, d\mathbb{P}$$

and $\int_E Y_T^n \, d\mathbb{P} = 0 \quad \forall E \in \mathcal{F}_T$. So $Y_T^n = 0$ in $L^1$ and $\mathbb{P}$-almost surely.

Now the sequence of stopping times, $\sigma_n$, increase to infinity (because $Y_T$ is finite $\mathbb{P}$-a.s.) and

$$\lim_n 2n Y_T^n = Y_T \quad \mathbb{P}.a.s.$$

So $Y_T = 0$. In other words the closed subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ generated by

$$\left\{ \int_0^T f(s)\, dW_s \; : \; \mathbb{E}\left(\int_0^T |f|^2\right) < \infty \right\}$$

is all of $\{X \in L^2(\Omega, \mathcal{F}, \mathbb{P}) : \mathbb{E}(X) = 0\}$.

(*) Let $S = \{E : \mathbb{P}(E) = \mathbb{Q}(E)\}$

(i) $\Omega \in S$, since $W_t^{-1}(\mathbb{R}) = \Omega$

(ii) If $E \in S$, $\mathbb{P}(\Omega \setminus E) = 1 - \mathbb{P}(E) = 1 - \mathbb{Q}(E)$ so $\Omega \setminus E \in S$.

(iii) If $E_n \in S$ and $E_i \cap E_j = \emptyset$ for $i \neq j$ then

$$\mathbb{P}\left(\bigcup_n E_n\right) = \sum_1^\infty \mathbb{P}(E_n) = \sum_1^\infty \mathbb{Q}(E_n) = \mathbb{Q}\left(\bigcup_n E_n\right)$$

so $\bigcup_n E_n \in S$ and $S$ is closed under monotone sequential limits and therefore under countable unions. So $S$ is a $\sigma$-field containing the generators of $\mathcal{F}_T$ and is itself inside of $\mathcal{F}$, so $S = \mathcal{F}_T$.

<u>Remark</u> I have not explicitly dealt with the completion or augmentation of the $\sigma$-field generated by $W$. Suffice to say that since $\mathbb{P}$ and $\mathbb{Q}$ agree on the $\sigma$-field generated by $W$, the completion and augmentation with respect to either will be the same.