The Black-Scholes Formula

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Introduction

These notes look at a number of ways of arriving at the Black - Scholes formula for the price of a European call option. It is assumed that the reader is familiar with the idea of an admissible self-financing portfolio, the definition of a European call option and elementary stochastic calculus. Each of the sections that follow arrive at the Black - Scholes formula in different ways. No doubt we could demonstrate that some of these are mathematically equivalent, but this is another project. The aim is give the reader the means to compare existing derivations of the result and to provide an editorial commentary which I hope is not too distracting.

Let us fix some notation: Our *risky* asset, a *stock*, has price S_t at time t and we consider it over the time interval $[0, T]$ during which its dynamics are given by

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s$$

for $t \in [0, T]$. At the same time our *risk-less* asset, a *bond*, has price B_t at time t and its dynamics are given by

$$B_t = 1 + \int_0^t r B_s ds$$

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for $t \in [0, T]$ where $r > 0$ is a constant taken to be the continuously compounded (risk-less) interest rate. Our European call option with strike price K is the right but not the obligation to buy the stock for the price K at time T. This right should come at a price and it was the achievement of Black and Scholes to give a rational price for this and other options. A key idea in all of this is *arbitrage*. We assume the reader is familiar with this idea and its mathematical formulation.

1 A conventional derivation

Suppose that we have written a European call option on the stock. At the outset we have the problem of what charge we should make for this derivative security and at expiry, time T, we have the problem of meeting our obligations. Now it is assumed throughout the life of the option there are no taxes, transaction costs, bid-offer spreads and that we may sell short either the stock or the bond and have full use of the proceeds immediately. Under these conditions it is possible to construct a portfolio of stock and bond which actually replicates the option value over $[0, T]$. To see how one can do this, consider forming a portfolio consisting of a European call option on the stock, a certain number of stocks, $\phi(t)$, and bonds, $\psi(t)$, at time t. We shall denote the value of the option at time t by $P(t, S_t)$ where $P(t, x)$ is a $C^{1,2}$ function of time, t, and its spatial variable, x. The reader should note that this assumption is rather strong in that it restricts the influences on the options value. This portfolio, whose value at time t we shall denote by $V(t)$, will be managed by altering the stock and bond amounts in a self - financing manner so that

$$V(t) = P(t, S_t) + \phi(t)S_t + \psi(t)B_t$$

and

$$V(t) = V(0)' + P(t, S_t) + \int_0^t \phi(s)dS_s + \int_0^t \psi(s)dB_s.$$

Here $V(0)' = V(0) - P(0, S_0)$. Now, using Ito's Lemma (the fundamental theorem of stochastic calculus) we can express the options value as follows;

$$P(t, S_t) = P(0, S_0) + \int_0^t \frac{\partial P(s, S_s)}{\partial s} ds + \int_0^t \frac{\partial P(s, S_s)}{\partial x} dS_s + \int_0^t \frac{\partial^2 P(s, S_s)}{\partial x^2} \frac{\sigma^2 S_s^2}{2} ds.$$

We observe that the integral with respect to the stock price can be expanded to

$$\int_0^t \frac{\partial P(s,S_s)}{\partial x} dS_s = \int_0^t \frac{\partial P(s,S_s)}{\partial x} \mu S_s ds + \int_0^t \frac{\partial P(s,S_s)}{\partial x} S_s \sigma dW_s.$$

Recalling that we can write $V(t)$ in its self - financing form we can also expand the integrals with respect to S and B appearing there to get

$$V(t) = V(0)' + P(t,S_t) + \int_0^t (\phi(s) S_s \mu + r\psi(s) B_s) ds + \int_0^t \phi(s) \sigma S_s dW_s.$$

To continue the analysis of this portfolio I need to combine together both the expanded form for $P(t,S_t)$ and the self financing form for $V(t)$ above. The limitations of space on the page demand that I do this in stages so let us first of all combine together all of the integrals with respect to "ds". These terms amount to

$$\int_0^t (\frac{\partial P(s,S_s)}{\partial s} + \frac{\partial P(s,S_s)}{\partial x} \mu S_s + \frac{1}{2} \frac{\partial^2 P(s,S_s)}{\partial x^2} \sigma^2 S_s^2 + \phi(s) S_s \mu + r\psi(s) B_s) ds$$

while the stochastic integral terms amount to

$$\int_0^t (\phi(s) S_s + \frac{\partial P(s,S_s)}{\partial x} S_s) \sigma dW_s.$$

So our portfolio can now be expressed as the sum of three terms,

1. its initial value, $V(0)$[1], involving the initial value of the option plus that of the self - financing portfolio of stock and bond,
2. the terms involving the integrals with respect to "ds",
3. the terms involving stochastic integration with respect to "dW".

Now that we have expressed the value of our portfolio in these terms we can take the next step which is to describe how we manage it (aside from the self - financing condition). What we seek is a portfolio that is *instantaneously risk-less* at each moment in time. Before we continue with the mathematics, we remark upon two aspects of this strategy. First of all what do we mean by a portfolio being risk-less? Second, why should we choose a risk-less portfolio at all? The first aspect seems simple enough, a portfolio is risk-less

[1]Observe that $V(0)'$ gets together with $P(0,S_0)$ to give $V(0)$

if its return is certain for the period in question. This raises a mathematical question; how do we characterise risk-less portfolios? The answer, according to Black and Scholes, is that the dynamics of its price, R_t, should be of the form

$$R_t = R_0 + \int_0^t T_s ds$$

for some process, T [2]. Notice that there is no term involving *stochastic* integration in the equation above. This must rank as the least satisfactory part of the argument presented by Black and Scholes and it led to 'a minor industry of getting the argument right'. The second aspect seems at first to be only reasonable; if one holds a portfolio there may be downside risk. It is natural, from at least one point of view, to want to avoid this risk. But how does this help with our management of the portfolio? An heuristic argument is that movement in the value of the option should be offset by a movement in the portfolio of stock and bond so that the entire portfolio (option plus stock plus bond) grows at a steady rate without any volatility. It turns out that this is sufficient for us to be able to determine a portfolio of stock and bond which tracks the value of the option exactly.

So let us return to our portfolio with value $V(t)$ given by the three items listed above. Given our remarks about risk-less portfolios, it is now clear what strategy we should adopt to render our portfolio risk-less, we must set

$$\phi(t) = -\frac{\partial P(t, S_t)}{\partial x}.$$

This eliminates the third item of our list and modifies the second to leave us with

$$V(t) = V(0) + \int_0^t (\frac{\partial P(s, S_s)}{\partial s} + \frac{1}{2}\frac{\partial^2 P(s, S_s)}{\partial x^2}\sigma^2 S_s^2 + r\psi(s)B_s)ds.$$

At this point the second key ingredient of the Black - Scholes argument comes into play. Our portfolio is now a 'risk-less' one. Therefore it should grow at exactly the same rate as the risk-less bond. The argument for this follows from the *assumption* that there are no arbitrage opportunities available to investors. Should our portfolio exhibit a return that is different from the risk free return then by selling one short (the worse!) and buying the other (the better) one can lock in a risk-less profit immediately (if you're cute) or later

[2] There is an annex to these notes that deals with this matter.

if you're conventional. This 'no-arbitrage' argument hides a mathematical difficulty addressed in the annex. What we now have is

$$V(t) = V(0) + \int_0^t V(s)rds$$

and, at the same time

$$V(t) = V(0) + \int_0^t (\frac{\partial P(s, S_s)}{\partial s} + \frac{1}{2}\frac{\partial^2 P(s, S_s)}{\partial x^2}\sigma^2 S_s^2 + r\psi(s)B_s)ds.$$

Putting these two parts together while noticing that the terms involving $\psi(s)$ cancel gives

$$\int_0^t (rP(s, S_s) - r\frac{\partial P(s, S_s)}{\partial x}S_s - \frac{\partial P(s, S_s)}{\partial s} - \frac{1}{2}\frac{\partial^2 P(s, S_s)}{\partial x^2}\sigma^2 S_s^2)ds = 0 \quad (1)$$

for every $t \in [0, T]$. From this it is implied that the integrand is zero so that

$$rP(s, S_s) - r\frac{\partial P(s, S_s)}{\partial x}S_s - \frac{\partial P(s, S_s)}{\partial s} - \frac{1}{2}\frac{\partial^2 P(s, S_s)}{\partial x^2}\sigma^2 S_s^2 = 0.$$

For $s \in [0, T]$. This suggests that the function $P(t, x)$ is a solution to the partial differential equation

$$rP(t, x) - r\frac{\partial P(t, x)}{\partial x}x - \frac{\partial P(t, x)}{\partial t} - \frac{1}{2}\frac{\partial^2 P(t, x)}{\partial x^2}\sigma^2 x^2 = 0. \quad (2)$$

I use the term "suggests" on purpose. The condition given in equation 1 and the path-wise properties of S_t need to be employed to justify the assertion that $P(t, x)$ does indeed satisfy the partial differential equation in a given region of (t, x)-space. Let us look at this point briefly. We can rewrite the equation in 1 to emphasise what is going on; let

$$g(t, x) = rP(t, x) - r\frac{\partial P(t, x)}{\partial x}x - \frac{\partial P(t, x)}{\partial t} - \frac{1}{2}\frac{\partial^2 P(t, x)}{\partial x^2}\sigma^2 x^2.$$

Then $g(t, x)$ is a continuous function of both variables. When we write $g(s, S_s(\omega))$, for $s \in [0, T]$, we are evaluating g on the graph of the *path* , $s \to S_s(\omega)$. For each continuous path of S then 1 holds. It follows that $g(t, x)$ is zero on these paths - differentiate the integral. Now, do the paths of S completely fill our (t, x)-space? The answer is yes. This follows from the fact that the paths of Brownian Motion include every continuous function (on $[0, T]$). As an exercise you might like to consider what path(s) Brownian Motion must follow in order that the path(s) of S should follow some prescribed function passing through a finite number of prescribed points of the (t, x) space. From these considerations we can deduce that P should satisfy the partial differential equation given in equation 2. This equation is called the Black - Scholes equation. We should observe at this point that we have not, so far, used the fact that $P(t, x)$ is intended to be the function of time and stock price which determines the value of the option. It follows that the Black - Scholes equation is a quite general equation for the value of a security whose value depends upon the stock price and time. When we specify precisely the nature of the security we will specify precise boundary conditions for the Black - Scholes partial differential equation and these will pick out the appropriate solution which determines the securitys price over time and the varying values of the stock. So, for example our option will have value at expiry given by, $(S_T - K)^+$ that is the boundary condition for P at time T will be $P(T, x) = (x - K)^+$. To specify the other boundary conditions for $P(t, x)$ we observe that the region on which $P(x, t)$ is defined is $[0, T] \times (0, \infty)$, a rectangle with base the interval $[0, T]$, and comprising the infinite strip *above* this in the $(t.x)$ plane. We have already specified its behaviour on the right hand side of the rectangle. On the bottom of this rectangle we assign the boundary condition $P(t, 0) = 0$, which is reasonable because as the stock price tends to zero the option becomes worthless. On the other hand, as the stock price tends to ∞ the value of the stock and the option become *relatively* the same, giving the boundary condition that $\frac{P(t,x)}{x} \to 1$ as $x \to \infty$.

We are now able to to specify a solution to the Black-Scholes p.d.e.. There are many treatments of this part of the subject which motivate the *choice* of the particular form of the solution. There is a careful and logical treatment in "*The Mathematics of Financial Derivatives*" by Dewynne, Howison and Wilmott, CUP. So, we shall take as our departure point the *claim* that the following function solves the Black - Scholes p.d.e. . First of all let

$$d_1(t, x) = \frac{ln(\frac{x}{K}) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

where K is our strike price, r the (constant) risk-free rate and σ the constant volatility of the stock price. We also define

$$d_2(t,x) = d_1(t,x) - \sigma\sqrt{t}.$$

Now let $N(t)$ denote the distribution function of the standard normal random variable, so

$$N(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{x^2}{2}}dx$$

Our claim is that

$$\nu(t,x) = xN(d_1(T-t,x)) - Ke^{-r(T-t)}N(d_2(T-t,x))$$

solves 2. Consider the values of ν as $x \to 0$. We see that for a fixed t the term $d_1(t,x)$ tends to $-\infty$ so that *both* $N(d_1)$ and $N(d_2)$ tend to zero and hence ν tends to zero also. In fact this shows (see later) that the partial derivative of ν with respect to x also converges to zero as x tends to zero, so we can "smoothly" assume that ν takes the value 0 on $(t,0)$ for $t \in [0,T]$. On the vertical line (T,x) the values of ν are, strictly speaking, undefined because the formula involves division by zero. but what we intend here is that the values that ν takes on this line should be the limit of the values for $t < T$. What we see is that the limiting value of $d_1(T-t,x)$ depends upon the relationship between x and K. If $x > K$ then $ln\frac{x}{K}$ is strictly greater than 0 and $d_1(T-t,x)$ tends to $+\infty$ as $t \to T$ as does d_2 so that $\nu(t,x) \to x-K$. If $x = K$ then both $d_1(T-t,x)$ and $d_2(T-t,x)$ tend to zero as $t \to T$ and, as $x = K$,

$$\nu(t,x) \longrightarrow xN(0) - 1 \cdot xN(0) = 0$$

as $t \to T$. If $x < K$ then $ln(\frac{x}{K}) < 0$ and as $t \to T$ both d_1 and d_2 tend to $-\infty$ so that both the terms involving the distribution function tend to zero and ν follows them. This shows that $\nu(t,x) \to (x-K)^+$ as $t \to T$ as required. The behaviour of ν as x gets large is identical with what we would expect of the option price. We turn now to showing that ν does indeed satisfy the equation 2.

To begin with

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}},$$

so that

$$\frac{\partial N}{\partial t}(d_2) \quad = \quad \frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1-\sigma\sqrt{T-t})^2}{2}} \cdot (\frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}) \tag{3}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{d_1^2}{2}}\, \frac{\partial d_1}{\partial t}\, e^{r(T-t)}\, \frac{x}{K} + \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}\, \frac{\sigma}{2\sqrt{T-t}}\, \frac{x}{K}\, e^{r(T-t)} \\
&= \frac{\partial N}{\partial t}(d_1) e^{r(T-t)} \frac{x}{K} + \Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}\, \frac{1}{x\sigma 2\sqrt{T-t}}\Big) \frac{\sigma^2 x^2}{K} e^{r(T-t)} \\
&= \frac{\partial N}{\partial t}(d_1) e^{r(T-t)} \frac{x}{K} + \frac{\partial N}{\partial x}(d_1) \frac{\sigma^2 x^2}{2K} e^{r(T-t)}
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
\frac{\partial \nu}{\partial t} &= x\frac{\partial N}{\partial t}(d_1) - Ke^{-r(T-t)} \frac{\partial N}{\partial t}(d_2) - rKN(d_2)e^{-r(T-t)} \qquad (4) \\
&= x\frac{\partial N}{\partial t}(d_1) - Ke^{-r(T-t)} \{\frac{\partial N}{\partial t}(d_1) e^{r(T-t)} \frac{x}{K} + \frac{\partial N}{\partial x}(d_1) \frac{\sigma^2 x^2}{2K} e^{r(T-t)}\} \\
&- rKN(d_2)e^{-r(T-t)} \\
&= -\frac{\partial N}{\partial x}(d_1) \frac{\sigma^2 x^2}{2} - rKN(d_2)e^{-r(T-t)}.
\end{aligned}
$$

We turn to consider $\frac{\partial \nu}{\partial x}$. First of all we note that $\frac{\partial d_1}{\partial x} = \frac{\partial d_2}{\partial x}$ and that

$$
\frac{\partial N}{\partial x}(d_2) = \frac{\partial N}{\partial x}(d_1 - \sigma\sqrt{T-t}) = e^{-\frac{d_1^2}{2}} e^{\sigma\sqrt{T-t}\, d_1} e^{-\frac{\sigma^2}{2}(T-t)} = e^{-\frac{d_1^2}{2}} \frac{x}{K} e^{r(T-t)}.
$$

So that

$$
\frac{\partial N}{\partial x}(d_2) = \frac{x}{K} e^{r(T-t)} \frac{\partial N}{\partial x}(d_1).
$$

We now have

$$
\begin{aligned}
\frac{\partial \nu}{\partial x}(t,x) &= N(d_1) + x\frac{\partial N}{\partial x}(d_1) - Ke^{-r(T-t)} \frac{\partial N}{\partial x}(d_2) \qquad (5) \\
&= N(d_1) + x\frac{\partial N}{\partial x}(d_1) - Ke^{-r(T-t)} \frac{x}{K} e^{r(T-t)} \frac{\partial N}{\partial x}(d_1) \\
&= N(d_1)
\end{aligned}
$$

.

This is rather interesting! Recall the discussion of how we arrived at a riskless portfolio, we chose our stock amounts to be $-\frac{\partial P}{\partial x}$. Perhaps we can interpret the terms appearing in the expression for the option price, P, in terms of our hedging strategy? In any event we immediately get

$$
\frac{\partial^2 \nu}{\partial x^2} = \frac{\partial N}{\partial x}(d_1).
$$

Now we can put the parts together

$$\begin{aligned}
\frac{\partial \nu}{\partial t} &= -\frac{\partial N}{\partial x}(d_1)\frac{\sigma^2 x^2}{2} - rKN(d_2)e^{-r(T-t)} \qquad (6) \\
\frac{\sigma^2 x^2}{2}\frac{\partial^2 \nu}{\partial x^2} &= \frac{\partial N}{\partial x}(d_1)\frac{\sigma^2 x^2}{2} \\
xr\frac{\partial \nu}{\partial x} &= xrN(d_1) \\
-r\nu &= -xrN(d_1) + rKe^{-r(T-t)}N(d_2)
\end{aligned}$$

Their sum is zero. We now have a price at time 0 for our option, it is

$$\nu(0, S_0) = S_0 N(d_1(T, S_0)) - Ke^{-r(T-t)}N(d_2(T, S_0)).$$

We can also arrive at a *hedging strategy* for the option. In fact this part of the treatment coincides with that in the next section, so we move to that now.

2 Merton's argument

We saw that there were some mathematical difficulties with the material of the last section. Armed with the knowledge supplied by that section, we define a stochastic process,

$$V(s) = \nu(s, S_s(\omega)).$$

Here ν is exactly the solution to the Black-Scholes equation 2 discussed in the last section and we have given the name V to this process because it is to be the *value process* for a portfolio, which we now construct. We define a trading strategy in the underlying stock, $\phi(s) = \frac{\partial \nu}{\partial x}(s, S_s(\omega))$. Notice that this differs, by a sign, from the strategy adopted in the previous section to make the portfolio there risk-less. We now define

$$\psi(s) = \frac{V(s) - \phi(s)S_s}{B(s)}$$

then we have (obviously)

$$V(s) = \phi(s)S_s + \psi(s)B(s).$$

Now by Ito's Lemma

$$\nu(s,S_s)=\nu(0,S_0)+\int_0^t \frac{\partial\nu}{\partial t}(s,S_s)ds+\int_0^t \frac{\partial\nu}{\partial x}(s,S_s)dS_s+\int_0^t \frac{1}{2}\frac{\partial^2\nu}{\partial x^2}(s,S_s)\sigma^2 S_s^2 ds.$$

But because ν satisfies the Black - Scholes p.d.e.,

$$\begin{aligned}\nu(s,S_s) &= \phi(s)S_s+\psi(s)B(s) \\ &= \frac{\partial\nu}{\partial x}(s,S_s)S_s+\psi(s)B(s) \\ &= \frac{\partial\nu}{\partial x}(s,S_s)S_s+\frac{1}{r}(\frac{\partial\nu}{\partial s}(s,S_s)+\frac{1}{2}\frac{\partial^2\nu}{\partial x^2}(s,S_s)\sigma^2 S_s^2)\end{aligned} \tag{7}$$

Forcing the conclusion

$$r\psi(s)B(s)=\frac{\partial\nu}{\partial s}(s,S_s)+\frac{1}{2}\frac{\partial^2\nu}{\partial x^2}(s,S_s)\sigma^2 S_s^2.$$

Substituting into the formula for ν given by Ito's Lemma above gives

$$V(s)=V(0)+\int_0^t \phi(s)dS_s+\int_0^t \psi(s)dB(s).$$

So our portfolio is self-financing and produces exactly the same cash flows as the option. If the price of this portfolio and the option differed, then by short selling one and buying the other one could achieve risk-less profitable arbitrage. Under the assumption that this is not possible the prices of these two instruments must coincide.

Martingale Methods

The assumptions governing this section are that the filtration is the P augmentation of that generated by the Brownian Motion, W , over $[0,T]$. The stock and bond prices are described by exactly the same stochastic equations as in our first section. We assume no transaction costs, margin requirements, taxes, etc and unlimited short selling of stock and bond are allowed.

A motivating case.

In order to motivate what follows we consider a special case. Suppose that our stock, S, follows the stochastic equation

$$S_t = S_0 + \int_0^t rS_s ds + \int_0^t \sigma S_s dW_s$$

so that S has drift equal to r, the risk free rate. Our option has payoff $(S_T - K)^+$ at time T and our task is to fashion a portfolio who's final value will be exactly the payoff of the option in all (or P-almost surely all) states of the world[3]. Now the random variable, S_T, is given by

$$S_T = S_0 e^{\sigma W_T + (r - \frac{\sigma^2}{2})T}$$

and since

$$\begin{aligned} E(e^{2\sigma W_T}) &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{+\infty} e^{2\sigma x} e^{-\frac{x^2}{2T}} dx \\ &= \frac{1}{\sqrt{2\pi T}} e^{-2T\sigma^2} \int_{-\infty}^{+\infty} e^{-\frac{(x-2T\sigma)^2}{2T}} dx \\ &< \infty \end{aligned} \tag{8}$$

then S_T is in $L^2(\Omega, \mathcal{F}, P)$ as is $S_T - K$ and its positive part. So that our target is an L^2 random variable. Note at this point that what we are seeking is an admissible portfolio which will replicate the option payoff. We are trying to prove that such a portfolio exists. So, it is not unreasonable to conclude that we might need to invoke an 'existence theorem' during the course of our proof. The existence theorem most familiar to us is the Martingale Representation Theorem.

The martingale representation theorem comes in many forms, one version states that in our situation, for any $L^2(\mathcal{F}_T)$ random variable, X, there is a constant, ν_0, and a progressively measurable process, $\rho(s)$, such that

$$M_t(X) = \nu_0 + \int_0^t \rho(s) dW_s,$$

[3]Remember, the set of all possible states of the world is interpreted as Ω. With this in mind, each $\omega \in \Omega$ corresponds to some state of the world over the time interval $[0, T]$

for $t \in [0,T]$. Taking $X = (S_T - K)^+$ the martingale representation theorem tells us that $\rho(s)$ exists such that

$$\begin{aligned} (S_T - K)^+ &= \nu_0 + \int_0^T \rho(s) dW_s \qquad (9) \\ &= \nu_0 + \int_0^T \rho(s) \frac{S_s}{S_s\sigma} \sigma dW_s \\ &= \nu_0 + \int_0^T (\frac{\rho(s)}{S_s}) r S_s ds + \int_0^T (\frac{\rho(s)}{S_s\sigma}) S_s \sigma dW_s - \int_0^T (\frac{\rho(s)}{B_s}) r B_s ds \\ &= \nu_0 + \int_0^T \phi(s) dS_s + \int_0^T \psi_s dB_s \end{aligned}$$

Here $\phi_s = \frac{\rho(s)}{S_s\sigma}$ and $\psi_s = -\frac{\rho(s)}{B_s}$. This shows our option payoff 'represented' as an initial endowment plus the gain from trading the strategy (ϕ, ψ). Of course we don't yet know if this is an admissible trading strategy! But this is a clue rather than a way forward. The deeper insight is that we have to take into account the *martingale* $(M_t(e^{-rT}(S_T - K)^+))$. We know that

$$M_t(e^{-rT}(S_T - K)^+) = \nu_0 + \int_0^t \rho(s) dW_s$$

We set $\phi_s = \frac{\rho_s e^{rs}}{\sigma S_s}$ and $\psi = M_t(e^{-rT}(S_T - K)^+) - \frac{\rho_s}{\sigma}$. Then

$$\begin{aligned} V(\phi,\psi)_t &= \phi_t S_t + \psi_t e^{rt} \\ &= \frac{\rho_t e^{rt}}{\sigma S_t} S_t + (M_t(e^{-rT}(S_T - K)^+) - \frac{\rho}{\sigma}) e^{rt} \\ &= M_t(e^{-rT}(S_T - K)^+) e^{rt}. \end{aligned}$$

Let $N_t = M_t(e^{-rT}(S_T - K)^+)$, we can use the product rule to write

$$\begin{aligned} N_t e^{rt} &= \nu_0 + \int_0^t N_s r e^{rs} ds + \int_0^t e^{rs} dN_s \\ &= \nu_0 + \int_0^t N_s r e^{rs} ds + \int_0^t e^{rs} \rho_s dW_s \\ &= \nu_0 + \int_0^t (N_s - \frac{\rho_s}{\sigma}) r e^{rs} ds + \int_0^t \frac{\rho_s r e^{rs}}{\sigma} ds + \int_0^t e^{rs} \rho_s dW_s \\ &= \nu_0 + \int_0^t (N_s - \frac{\rho_s}{\sigma}) r e^{rs} ds + \int_0^t \frac{\rho_s r e^{rs} S_s}{S_s\sigma} ds + \int_0^t e^{rs} \rho_s \frac{\sigma S_s}{\sigma S_s} dW_s \end{aligned}$$

$$
\begin{aligned}
&= \nu_0 + \int_0^t \psi_s dB_s + \int_0^t \phi_s r S_s ds + \int_0^t \phi_s S_s \sigma dW_s \\
&= \nu_0 + \int_0^t \psi_s dB_s + \int_0^t \phi_s dS_s
\end{aligned}
$$

So our portfolio is self-financing and because the portfolio tracks $e^{rt}N_t$ it is almost surely non-negative and hence admissible.. From this we get immediately that

$$(S_T - K)^+ = \nu_0 + \int_0^T \psi_s dB_s + \int_0^T \phi_s dS_s$$

which displays the option value as the value of the self-financing portfolio (ϕ, ψ) at time T. But the equations above tell us more. Write,

$$V_t = \nu_0 + \int_0^t \psi_s dB_s + \int_0^t \phi_s dS_s,$$

the value process of the strategy (ϕ, ψ). Then we have

$$e^{-rt}V_t = N_t.$$

So the discounted value of this portfolio is the martingale (N_t) and, in particular, its time zero value is

$$\nu = V_0 = N_0 = M_0(e^{-rT}(S_T - K)^+) = E(e^{-rT}(S_T - K)^+).$$

Because this portfolio replicates the final value of the option there is a standard arbitrage argument [4] which concludes that the initial value of the portfolio and the option must coincide. One could summarise the analysis above by saying that the martingale representation theorem provides the trading strategy completely (ghastly pun!) including the initial capital required.

Loose ends: The discounted portfolio value is a martingale, but on the face of it doesn't look like a martingale because it has a drift term. But refer back to our characterisation of self-financing portfolios. A portfolio is self financing if and only if its discounted value is an initial amount plus the integral of the stock strategy with respect to the discounted stock value. But in this specific case the discounted stock value is

$$\tilde{S} = e^{-rt}S_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2}t}$$

[4] It is a common practice to make the following argument at this stage:
The cost at time zero of the option cannot depart from that of the trading strategy, (ϕ, ψ), described above. For if it did, then one could sell (short) the greater and buy the lesser to lock in a risk-less profit, one's obligations being met at time T by the lesser instrument.

You can use Ito's Lemma to prove this is a(n) (exponential) martingale. Since it is a martingale the stochastic integral with respect to $\tilde{S}$ is also a martingale. Another observation is that the actual option value played no part whatsoever in this analysis. We could have replaced $(S_T - K)^+$ with any non-negative $\mathcal{F}_T$ measurable L^2 random variable. Finally, let us refer back to our equations at the head of page 13. We pick up the chain of equalities and part way through rewrite them:

$$
\begin{aligned}
V_t &= N_t e^{rt} \ldots\ldots\ldots\ldots \\
&= \nu_0 + \int_0^t (N_s - \frac{\rho_s}{\sigma}) r e^{rs} ds + \int_0^t \frac{\rho_s r e^{rs}}{\sigma} ds + \int_0^t e^{rs} \rho_s dW_s \\
&= \nu_0 + \int_0^t (N_s e^{rs} - e^{rs} \frac{\rho_s}{\sigma} \frac{S_s}{S_s}) r ds + \int_0^t \frac{\rho_s r e^{rs} S_s}{S_s \sigma} ds + \int_0^t e^{rs} \rho_s \frac{\sigma S_s}{\sigma S_s} dW_s \\
&= \nu_0 + \int_0^t (V_s - \phi_s S_s) r ds + \int_0^t \phi_s dS_s
\end{aligned}
$$

So here we have our self-financing portfolio displayed in dynamic form showing that at time s it achieves V_s by holding ϕ items of S leveraged by $V_s - \phi_s S_s$ bonds. So ψ is determined by V and the current stock holding ϕS. Of course, as we have seen, the discounted value of V may be expressed as a constant plus the integral of ϕ with respect to the discounted value of S. So ψ is 'determined by' ϕ and S (and r).

In the specific case examined above we first used the martingale representation theorem and from this obtained a self-financing portfolio. If you try to prosecute *this* argument on a stock with drift $\mu \neq r$ a difficulty arises. To get the portfolio to track $e^{rt} N_t$ you are forced into the choices we made for ϕ and ψ. But then this portfolio is not self financing. The argument goes awry at line three of the equations at the top of page 13 and thereafter. How can we remedy this? The answer is provided by Girsanov's Theorem.

While this is basically true I want to examine the argument in some detail. The problem I am trying to address here is how the mathematics and the finance interact. For example, one justification for the argument above is; if on any *path* of S the values of the option and the portfolio differ then by short selling one and buying the other an arbitrage opportunity occurs. This argument makes especially good sense if we imagine that we are watching the particular price evolutions unfold and we act in the way described. Of course this does not *prove*[5] that the assumption of *no-arbitrage* entails the agreement of the option price and the portfolio value. Recall that an arbitrage opportunity is an admissible trading strategy, Φ, such that the portfolio value, $V_t(\Phi)$, satisfies

$$V_0(\Phi) = 0 \quad \text{but that} \quad E(V_T(\Phi)) > 0$$

The argument rehearsed above seems to allow the existence of arbitrage because of a state of affairs on a single path of S at a single moment of time, t. But this does *not* lead to the mathematical expression for arbitrage above. This is because what happens on one particular path refers to the singleton, $\{\omega\}$ which is a set of zero P measure and is therefore irrelevant. What this demonstrates is that one cannot pass seamlessly from the financial arguments to the mathematics.

Arbitrage

Before we proceed further let us consider whether or not an arbitrage opportunity can exist in our present situation. First of all our discounted stock price is a martingale under the measure P. Note, also, that an admissible (self-financing portfolio), $\Phi = (\phi, \psi)$ must satisfy

$$V_t(\Phi) = V_0(\Phi) + \int_0^t \phi(s) dS_s$$

for $t \in [0, T]$. So our discounted admissible portfolio is a martingale under P too because it is a constant (random variable) plus a stochastic integral with respect to S. It follows that $E(V_T(\Phi)) = E(V_0(\Phi)) = V_0(\Phi)$. So the

[5] It is worth pointing out here that up to this point we know very little! For example we have no reason to believe (yet) that the value of the portfolio and that of the option should coincide at every time t.

terminal expectation and the initial *value* must agree[6]. It follows that we cannot have the initial value zero while the terminal expectation is positive, that is, we cannot have arbitrage. But notice here that this refers only to the model consisting of a stock and a bond. However, since the option payoff is replicable by a portfolio of stock and bond then any portfolio consisting of bond, stock and option is equivalent to a portfolio of stock and bond *so long as the portfolio which replicates the payoff of the option also replicates the option value throughout* $[0, T]$. Since admissible 'stock-bond' portfolios are free of arbitrage this would mean that portfolios including options will also be free of arbitrage.

We will return to this point later but first we note that we can say more about the relationship between V_t and C_t if we assume that the extended model, consisting of the option, the stock and the bond is free from arbitrage. Indeed let the value of the option at time t be $C(t)$. Suppose also that at some time $t \in [0, T)$ we have $C(t) - V_t(\Phi)$ is *not* zero P almost surely. Then at least one of $(C(t)-V_t(\Phi))^+$ and $(C(t)-V_t(\Phi))^-$ must have strictly positive expectation. For the sake of argument consider the case $(C(t) - V_t(\Phi))^+$ has strictly positive expectation. Let E_t be the $\mathcal{F}_t$ set where $(C(t) - V_t(\Phi))^+$ is non-zero. Now we define a trading strategy: If $s < t$ then we hold no stocks and no bonds. At time t we hold nothing if $\omega \in \Omega \setminus E_t$ while if $\omega \in E_t$ we sell the option[7], buy the portfolio and, further, put $C(t) - V_t(\Phi)(\omega)$ into the bond. The portfolio is managed in a self-financing manner to replicate the value of the option at time T on the set E_t. The portfolio meets ones obligations while the there remains the random amount

$$(C(t) - V_t(\Phi)I_{E_t}e^{r(T-t)}$$

Which has expectation

$$E((C(t) - V_t(\Phi)I_{E_t}e^{r(T-t)}) = e^{r(T-t)}E((C(t) - V_t(\Phi))^+) > 0.$$

This shows that the option value and the portfolio value must agree P almost surely at each time, $t \in [0, T]$. Now we know that our portfolio has P almost surely continuous paths and if we assume the same of the option then we can convert the property that V_t and C_t should agree P almost surely at

[6]Notice that this expectation being a number depends upon the time zero σ-field being simply $\{\emptyset, \Omega\}$

[7]In short, we write an option on E_t and invest the proceeds in the portfolio with the remainder into the bond

each time t into the condition that their paths will *agree* for P almost every $\omega \in \Omega$. In other words, (C_t) and (V_t) are, up to indistinguishability, the same stochastic process.
So far then we have that there can be no arbitrage in the stock and bond model. If the stock, bond and option model has no arbitrage then the replicating portfolio and the option value must 'agree' and hence the time zero value of the replicating portfolio must agree with the option price at that time. But this evades the *mathematical* question; is the model consisting of stock, bond and an option on the stock, free from arbitrage? Notice that the problem here is that (previously) the price of the option has been inferred from a combination of no-arbitrage and the existence of a portfolio which replicates its payoff at time T. So is 'no-arbitrage' an extra assumption which we bolt on to our mathematical theory? If so, we need to be sure that it is mathematically consistent[8]! In order to keep to our theme we shall steer around this point and not discuss it. We shall *assume* hereafter that there is no portfolio consisting of finitely many assets, stock, bond, and derived securities such as options, which when managed in an admissible fashion can generate an arbitrage opportunity as we move on past our special case.

[8]That is, its assumption does not lead to contradictions

3 A more general case

If we look back at our special case. We can identify several features of the situation which made the problem tractable.

1. The discounted stock price was a martingale under the measure P.
2. The rate of return of S, the drift, was equal to r the risk-free rate.
3. The martingale representation theorem showed us how to construct a replicating portfolio for any (L^2) contingent claim.
4. The absence of arbitrage determined the value of the claim at each time, $t \in [0, T]$.

If we consider the case where our stock price, S, has a drift term then we cannot apply the previous analysis because our stock price is no longer a martingale under P. However, by using Girsanov's change of measure theorem we can 'introduce a probability measure under which the (discounted) stock price is a martingale and then run an analysis which parallels that of the previous section to obtain both a price and a replicating strategy for our option.

We will work with discounted asset prices now (there is a practical and a theoretical advantage to this). So let Z denote the discounted stock price. It is easy to see that [9]

$$Z_t = \frac{S_t}{B(t)} = Z_0 + \int_0^t (\mu - r) Z_s ds + \int_0^t \sigma Z_s dW_s.$$

Just as previously, we are interested in forming a portfolio, $(\psi(t), \phi(t))$, which replicates the option value. But as we are working with discounted values we look for a portfolio such that

$$\psi(T) + \phi(T) Z_T = \frac{(S_T - K)^+}{B(T)}.$$

For each $t \in [0, T]$, we let $V^*(t) = \psi(t) + \phi(t) Z_t$. This is the *discounted* portfolio value (function, for the trading strategy (ψ, ϕ)). In order that the

[9]Use the formula for the product of semimartingales or Ito's Lemma and the function $f(t, x) = e^{-rt}x$.

trading strategy be self-financing we also require that at each time, t, [10],

$$V^*(t) = V_0 + \int_0^t \phi(s) dZ_s.$$

So, if the strategy ϕ is given then ψ is obtained by setting

$$\psi(t) = V_0 + \int_0^t \phi(s) dZ_s - \phi(t) Z_t.$$

This calculates ψ in terms of the current observables; the initial endowment, the value of the current (discounted) stock holding and the gain due to trading with ϕ in the discounted stock, Z. Consider again for the moment the stochastic equation for the discounted stock price;

$$Z_t = \frac{S_t}{B(t)} = Z_0 + \int_0^t (\mu - r) Z_s ds + \int_0^t \sigma Z_s dW_s.$$

We can rewrite this as

$$Z_t = \frac{S_t}{B(t)} = Z_0 + \int_0^t Z_s \sigma d(\frac{(\mu - r)}{\sigma} s + W_s).$$

This is not just a piece of notational convenience, the integral of (Z_s) with respect to $\frac{\mu - r}{\sigma} s + W_s$ makes perfectly good sense; see the notes on integration with respect to semi-martingales. However, we have written this equation in this way for a good reason which we hope now to make clear. Girsanov's theorem tells us that if we introduce the *exponential martingale*,

$$\eta_t = e^{\int_0^t \frac{\mu - r}{\sigma} dW_s - \frac{1}{2} \int_0^t (\frac{\mu - r}{\sigma})^2 ds}$$

for $t \in [0, T]$, then the set function for $E \in \mathcal{F}_T$ given by

$$Q(E) = \int_E \eta_T dP$$

is a probability measure equivalent to P under which $(W_t + \frac{\mu - r}{\sigma} t)$ is a martingale. Let us verify some of these details before we move on.

[10] If you are unfamiliar with working with discounted quantities I have some hand written notes which deal with converting the non-discounted formulation into discounted form and vice-versa.

Lemma 1 *The exponential, η_t, satisfies the stochastic equation*

$$\eta_t = 1 + \int_0^t \eta_s(\frac{\mu - r}{\sigma})dW_s$$

Proof 1 *Apply Ito's Lemma to the function e^x and the semimartingale*

$$\frac{\mu - r}{\sigma}W_t - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 t.$$

This yields the equation given.

From this lemma we see immediately that (η_t) is indeed a martingale. But by its definition it is strictly positive and in particular one can check that η_T is integrable and P almost surely positive. From the theory of the abstract Lebesgue integral and the fact that stochastic integrals with respect to martingales have zero expectation, it follows that the set function, $Q(E)$, for $E \in \mathcal{F}_T$, is a probability measure on $\mathcal{F}_T$. Moreover if $P(E) = 0$ then $Q(E) = 0$. On the other hand if $Q(E) = 0$ then the random variable $I_E \eta_T$ must be zero P almost surely while at the same time η_T is strictly positive, P almost surely. So E must be a set of P measure zero (if it were not the integral of $I_E \eta_T$ would be strictly positive). So Q is a probability measure on $(\Omega, \mathcal{F}_T, P)$ equivalent to P. To see that $(W_t + \frac{\mu - r}{\sigma}t)$ is a martingale under Q you need to retrace some of the steps of the proof of Girsanov's theorem. See the proof of Girsanov's theorem in your notes for this. Moving on, we now see the reason for writing our discounted stock value as

$$Z_t = \frac{S_t}{B(t)} = Z_0 + \int_0^t Z_s \sigma d(\frac{(\mu - r)}{\sigma}s + W_s).$$

It is now clear that Z is expressed as a stochastic integral with respect to a Q Brownian motion! So, the discounted stock price follows a martingale so far as Q is concerned. Indeed if we write

$$W_t^* = W_t + \frac{\mu - r}{\sigma}t$$

then W^* is a Q Brownian motion and

$$Z_t = Z_0 + \int_0^t Z_s \sigma dW_s^*$$

which can be solved to give

$$Z_t = Z_0 e^{\sigma W_t^* - \frac{\sigma^2}{2} t}.$$

And the stochastic equation for S is then

$$S_t = S_0 + \int_0^t r S_s ds + \int_0^t \sigma S_s dW_s^*.$$

This shows that under the *martingale measure*, Q, the stock follows a (Q) logarithmically normal price evolution with mean rate of return equal to the risk-free rate, r.

We now consider whether or not our option is an *attainable claim*. The answer is in the affirmative. Recall that a claim is attainable if and only if there is a self financing admissible portfolio, $\Phi = (\psi, \phi)$, such that

$$V_T^*(\Phi) = \frac{(S_T - K)^+}{B(T)}.$$

Now, so far as the measure Q is concerned,

$$S_T = S_0 e^{\sigma W_T^* + (r - \frac{\sigma^2}{2})T}$$

and W^* is a Q Brownian motion. A computation (left to the reader) shows that S_T lies in $L^2(\Omega, \mathcal{F}_T, Q)$ and that, therefore $S_T - K$ and $\frac{(S_T-K)^+}{B(T)}$ lies in $L^2(\Omega, \mathcal{F}_T, Q)$ [11]. We now apply the martingale representation theorem for Q Brownian Martingales. There is a constant, ν_0, and a progressively measurable process ρ with

$$E(\int_0^T \rho_s^2 ds) < \infty$$

and

$$\frac{(S_T - K)^+}{B(T)} = \nu_0 + \int_0^T \rho_s dW_s^*.$$

Arguing much as before we write

$$\int_0^T \rho_s dW_s^* = \int_0^T \frac{\rho_s}{\sigma Z_s} \sigma Z_s dW_s^*$$

[11]It is quite straightforward to show that $Q\{S_T \leq \alpha\} = Q\{W_T^* \leq h(\alpha)\}$ for some function, h of α This gives you an integral which is a function of α. Differentiate to get the Q density for S_T and verify that it has finite second moment

and rename $\frac{\rho_s}{\sigma Z_s}$ to be ϕ_s. We have seen above how to create the (discounted) self-financing portfolio that attains $\frac{(S_T-K)^+}{B(T)}$. It only remains to observe that the time zero price of the option must therefore be

$$\nu_0 = E^Q(\frac{(S_T-K)^+}{B(T)},$$

while for $t \in [0,T]$ the time t price of the option, C_t, say, is identical with the value of the replicating portfolio. As we have seen, the value of the discounted portfolio, V_t^* is a Q martingale so that

$$C_t = V_t(\Phi) = B(t)V_t^*(\Phi) = B(t)E_t^Q(V_T^*(\Phi)) = B(t)E_t^Q(\frac{(S_T-K)^+}{B(T)}).$$

Let $G = \{\omega : S_T(\omega) > K\}$. Then $(S_T-K)^+ = (S_T-K)I_G$. We note also that

$$(S_T-K)^+I_G = (S_te^{\sigma(W_T^*-W_t^*)+(r-\frac{\sigma^2}{2})(T-t)})I_G - KI_G$$

and that

$$\begin{aligned} E_t^Q((S_te^{\sigma(W_T^*-W_t^*)+(r-\frac{\sigma^2}{2})(T-t)})I_G - KI_G) &= S_tE_t^Q(e^{\sigma(W_T^*-W_t^*)+(r-\frac{\sigma^2}{2})(T-t)})I_G) \\ &- E_t^Q(KI_G). \end{aligned} \quad (10)$$

First of all we will compute the time zero price of the option and then argue further to get the time t price. So we consider first of all

$$\begin{aligned} B(0)E_0^Q(\frac{KI_G}{B(T)}) &= Ke^{-rT}Q(\{S_T > K\}) \\ &= Ke^{-rT}Q\{-\sigma W_T^* < log(\frac{S_0}{K}) + (r-\frac{\sigma^2}{2})T\} \\ &= Ke^{-rT}Q\{\frac{-W_T^*}{\sqrt{T}} < \frac{log(\frac{S_0}{K}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\} \\ &= Ke^{-rT}N(d(S_0,T)) \end{aligned}$$

Observe that $\frac{-W_T^*}{\sqrt{T}}$ is a standard normal random variable [12] and here we have written $N(x)$ for the normal distribution function while

$$d(x,t) = \frac{log(\frac{x}{K}) + (r-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}.$$

[12] Because $-W^*$ is a Brownian motion under Q

We turn now to consider the term

$$B(0)E_0^Q(\frac{S_T}{B(T)}I_G) = E^Q(Z_T I_G).$$

We are going to evaluate this by means of another change of measure, this idea has been used often in mathematical finance and it amounts to viewing the situation in a particular way - more of which later. Let R denote the measure on $\mathcal{F}_T$ given by

$$R(H) = \int_\Omega e^{\sigma W_T^* - \frac{\sigma^2}{2}T} I_H dQ$$

Now the process, $(W^0 = W_t^* - \sigma t)$, is a standard Brownian Motion with respect to the measure R. This follows directly from Girsanov's theorem. Moreover, rewriting

$$Z_T = Z_0 e^{\sigma W_T^* - \frac{\sigma^2}{2}T}$$

in terms of W^0 gives

$$Z_T = S_0 e^{\sigma W_T^0 + \frac{\sigma^2}{2}T}.$$

We can now write

$$\begin{aligned} E^Q(Z_T I_G) &= S_0 R(\{Z_T > \frac{K}{B(T)}\} \\ &= S_0 R\{-\sigma W_T^0 < log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T\} \end{aligned}$$

Following much the same argument as we employed above for the other term we find that

$$E^Q(Z_T IG) = S_0 N(d(S_0, T) + \sigma\sqrt{T}).$$

So our formula for the valuation of the call at time 0 is

$$C_0 = S_0 N(d(S_0, T)) - e^{-rT} K N(d(S_0, T) + \sigma\sqrt{T}).$$

The call value is expressed in terms of the time to expiry, the current value of the stock and functions of these in combination with the risk-free rate and stock volatility.

This gives the valuation for the call at time zero. If we consider what happens at time $t \in (0, T]$ when S takes a specific value, x, say, then our formula above may be modified to give the valuation

$$C(x)_t = xN(d(x, T-t)) - e^{-r(T-t)} K N(d(x, T-t) + \sigma\sqrt{T-t}).$$

It is a short step to see that this amounts to

$$C_t = S_t N(d(S_t, T-t)) - e^{-r(T-t)} K N(d(S_t, T-t) + \sigma\sqrt{T-t}).$$

Of course this is not a rigorous mathematical argument. In order to provide that we need to return to equation 10 above. Recall that we had

$$\begin{aligned} E_t^Q((S_t e^{\sigma(W_T^* - W_t^*) + (r - \frac{\sigma^2}{2})(T-t)}) I_G - K I_G) &= S_t E_t^Q(e^{\sigma(W_T^* - W_t^*) + (r - \frac{\sigma^2}{2})(T-t)}) I_G) \\ &- E_t^Q(K I_G). \end{aligned} \tag{11}$$

In order to deal with these terms we must bring in some results that we did not discuss in the course. I shall deal with these matters in subsequent sections of these notes which are in preparation. I hope to let you have them soon.

A Riskless Portfolio

This section is included in order to highlight the difficulties discussed in our first section. We investigate the possibility of holding a risk-free portfolio consisting of stock and the option. The argument presented in the annex, 'The Numeraire' shows that any risk-free portfolio, that is, a portfolio whose wealth process is a process of finite variation, must follow the same price evolution as the 'numeraire', in our case that of the bond B. So let us assume, as we have before, that the price of our option is given by *some* suitably smooth function $\nu(t, x)$ defined on $[0, T] \times (0, \infty)$ [13] such that $\nu(T, x) = (x-k)^+$. We have assumed then that at time $t \in [0, T]$ we have $C_t = \nu(t, S_t)$. We are going to adopt a trading strategy that consists of having sold one option we hedge by buying stock. So let ϕ denote the stock amounts and ψ the amount of option held. Denoting the strategy by, $\Phi = (\phi, \psi)$, we now *specify* $\Phi = (\frac{\partial \nu}{\partial x}(t, S_t), -1)$. This portfolio has value V_t, at time t, given by

$$V_t = \frac{\partial \nu}{\partial x}(t, S_t) S_t - C_t$$

using our assumption about the option value

$$V_t = \frac{\partial \nu}{\partial x}(t, S_t) S_t - \nu(t, S_t).$$

[13] What we hope to do is to show that this function ν is identical with that defined in our first section

Assuming that the trading strategy Φ is self-financing then

$$V_t = V_0 + \int_0^t \frac{\partial \nu}{\partial x}(t, S_t) dS_t + \int_0^t (-1) d\nu(t, S_t).$$

Now Ito's formula tells us how to integrate with respect to $\nu(t, S_t)$. Indeed

$$\begin{aligned} \int_0^t (-1) d\nu(s, S_s) &= \int_0^t (-1)(\mu S_s \frac{\partial \nu}{\partial x}(s, S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{\partial \nu}{\partial t}(s, S_s)) ds \\ &+ \int_0^t (-1)\sigma S_s \frac{\partial \nu}{\partial t}(s, S_s) dW_s \end{aligned} \tag{12}$$

putting this into our expression for V and writing the integral with respect to S_t fully gives us

$$V_t = V_0 + \int_0^t (-1)(\frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{\partial \nu}{\partial t}(s, S_s)) ds.$$

So our portfolio is risk-less! It follows from the argument presented in the annex, 'The Numeraire', that V must also satisfy the equation

$$V_t + V_0 + \int_0^t r V_s ds.$$

Now we can substitute for V_s directly in this equation and then equate this with the expression for V that we obtained immediately prior to this one. This gives

$$\int_0^t r S_s \frac{\partial \nu}{\partial x}(s, S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{\partial \nu}{\partial t}(s, S_s) - r\nu(s, S_s) ds = 0$$

for every $t \in [0, T]$. We are back upon familiar ground. From the arguments employed in our first section we deduce that ν must satisfy the Black-Scholes p.d.e. ;

$$rx \frac{\partial \nu}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \nu}{\partial x^2}(t, x) + \frac{\partial \nu}{\partial t}(t, x) - r\nu(t, x) = 0.$$

This deduction depends upon our assumption that the portfolio is self-financing. However, this proves not to be true! Recall that

$$V_t = \frac{\partial \nu}{\partial x}(t, S_t) S_t - \nu(t, S_t)$$

Using Ito's lemma on each part of the right side of this expression gives

$$\begin{aligned}\frac{\partial \nu}{\partial x}(t, S_t) &= \frac{\partial \nu}{\partial x}(0, S_0) + \int_0^t (\mu S_s \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^3 \nu}{\partial x^3}(s, S_s) \\ &+ \frac{\partial^2 \nu}{\partial t \partial x}(s, S_s))ds + \int_0^t \sigma S_s \frac{\partial^2 \nu}{\partial x^2}(s, S_s) dW_s\end{aligned}$$

and

$$\begin{aligned}\nu(t, S_t) &= \nu(0, S_0) + \int_0^t (\mu S_s \frac{\partial \nu}{\partial x}(s, S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{\partial \nu}{\partial t}(s, S_s))ds \\ &+ \int_0^t \sigma S_s \frac{\partial \nu}{\partial x}(s, S_s) dW_s.\end{aligned}$$

We can compute the product $S_t \frac{\partial \nu}{\partial x}(t, S_t)$ explicitly using the formula for the product of semi-martingales, however it will be a little more tidy if we don't calculate this product explicitly [14]. Now we can write two forms for V. First of all, if the strategy ϕ is self-financing then

$$\begin{aligned}V_t &= V_0 + \int_0^t \frac{\partial \nu}{\partial x}(s, S_s) dS_s + \int_0^t (-1) d\nu_s \\ &= V_0 + \int_0^t \frac{\partial \nu}{\partial x}(s, S_s) dS_s + (\nu_0 - \nu_t) \\ &= \frac{\partial \nu}{\partial x}(0, S_0) S_0 + \int_0^t \frac{\partial \nu}{\partial x}(s, S_s) dS_s - \nu_t\end{aligned}$$

while at the same time, using the product of semi-martingales formula gives

$$V_t = \frac{\partial \nu}{\partial x}(0, S_0) S_0 + \int_0^t \frac{\partial \nu}{\partial x}(s, S_s) dS_s + \int_0^t S_s d(\frac{\partial \nu}{\partial x})_s + < \frac{\partial \nu}{\partial x}, S >_t - \nu_t.$$

For these two expressions for V to be equal we must have

$$\int_0^t S_s d(\frac{\partial \nu}{\partial x})_s + < \frac{\partial \nu}{\partial x}, S >_t = 0.$$

Using our expression for $\frac{\partial \nu}{\partial x}$ derived (above) from Ito's lemma we get

$$\begin{aligned}\int_0^t S_s d(\frac{\partial \nu}{\partial x})_s &= \int_0^t (\mu S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{1}{2}\sigma^2 S_s^3 \frac{\partial^3 \nu}{\partial x^3}(s, S_s) \\ &+ S_s \frac{\partial^2 \nu}{\partial t \partial x}(s, S_s))ds + \int_0^t \sigma S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) dW_s \qquad (13)\end{aligned}$$

[14] There is a large amount of hindsight operating here.

and using the bilinearity of the cross variation gives

$$< \frac{\partial \nu}{\partial x}, S >= \int_0^t \sigma^2 S_s^2 \frac{\partial^2 \nu}{\partial x^2} ds.$$

Concentrate your attention on the term

$$\mu S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{1}{2}\sigma^2 S_s^3 \frac{\partial^3 \nu}{\partial x^3}(s, S_s) + S_s \frac{\partial^2 \nu}{\partial t \partial x}(s, S_s).$$

We are going to obtain a different form for this term. We have already shown that (assuming our portfolio is self-financing) our function ν must satisfy the Black-Scholes p.d.e. . Recall that this means

$$rx\frac{\partial \nu}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \nu}{\partial x^2}(t, x) + \frac{\partial \nu}{\partial t}(t, x) - r\nu(t, x) = 0.$$

Now differentiate this equation with respect to x. Assuming that the mixed partial derivatives are continuous the order of differentiation is immaterial. The result is,

$$rx\frac{\partial^2 \nu}{\partial x^2}(t, x) + \frac{1}{2}\sigma^2 2x \frac{\partial^2 \nu}{\partial x^2}(t, x) + \frac{\partial^2 \nu}{\partial x \partial t}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^3 \nu}{\partial x^3}(t, x) = 0.$$

Now by reaaranging this and multiplying by x we get

$$rx^2\frac{\partial^2 \nu}{\partial x^2}(t, x)+\sigma^2 x^2 \frac{\partial^2 \nu}{\partial x^2}(t, x)+x\frac{\partial^2 \nu}{\partial x \partial t}(t, x)+\frac{1}{2}\sigma^2 x^3 \frac{\partial^3 \nu}{\partial x^3}(t, x) = -\sigma^2 x^2 \frac{\partial^2 \nu}{\partial x^2}(t, x).$$

Adding in some (carefully chosen) terms we get that

$$\mu x^2\frac{\partial^2 \nu}{\partial x^2}(t, x)+x\frac{\partial^2 \nu}{\partial x \partial t}(t, x)+\frac{1}{2}\sigma^2 x^3 \frac{\partial^3 \nu}{\partial x^3}(t, x) = (\mu-r)x^2\frac{\partial^2 \nu}{\partial x^2}(t, x)-\sigma^2 x^2 \frac{\partial^2 \nu}{\partial x^2}(t, x).$$

Now substitute S_s for x and s for t in the arguments of the functions and we see that we have arrived at the different form for the terms

$$\mu S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) + \frac{1}{2}\sigma^2 S_s^3 \frac{\partial^3 \nu}{\partial x^3}(s, S_s) + S_s \frac{\partial^2 \nu}{\partial t \partial x}(s, S_s).$$

Substituting for these terms in equation 13 and combining with the equation for the cross variation following equation 13 leads to cancellation of the cross variation term and leaves us with

$$\int_0^t \sigma S_s^2 \frac{\partial^2 \nu}{\partial x^2}(s, S_s) d(W_s + \frac{(\mu - r)}{\sigma} s) = 0.$$

Now under the equivalent martingale measure, Q, the process $(W_t + \frac{(\mu-r)}{\sigma}t)$ is a Brownian Motion. For the integral in the last equation to be zero would have to be true that the second partial derivative of ν with respect to x is zero. This is not true! We must conclude that our portfolio is not self-financing.

4 Utility Functions

I have to give a brief summary of the relevant features of *Utility Functions* before moving to a further derivation of the Black-Scholes formula. First of all, imagine that one has two investment opportunities, A and B, with exactly the same initial investment X_0. If the outcome of each of A and B is certain, and we prefer more wealth than less, then it is easy to distinguish between the alternatives; we simply choose the larger of the two. But what if the outcomes are *random*? In this case the outcomes are described by random variables which we denote by A and B. Since these are simply functions defined on some probability space we cannot assert that one will be larger than the other [15]. For one state of the world, that is, $\omega_1 \in Omega$, we might have the outcome $A(\omega_1) > B(\omega_1)$ while for $\omega_2 \in \Omega$ with $\omega_2 \neq \omega_1$ it is perfectly possible to have, $B(\omega_2) > A(\omega_2)$. So how can we rank the outcome of these two investment possibilities? This question has been analysed at a fundamental level and the answer arrived at is that investor will use *Utility Functions* to rank the outcomes of (random) investment processes. A utility function is a real valued function of a real variable, U, say, which satisfies certain properties. First among these is that U be an increasing and continuous function. The way in which a utility function is used to distinguish between A and B is to compare the values of $E(U(A))$ and $E(U(B))$, where E denotes the mathematical expectation. In principle, any increasing and continuous function can be a utility function. Each investor must determine their utility function from their disposition toward risk. Investment Companies use questionnaires to determine an investors utility function. They can then advise investors in a systematic manner. We will pass over this important area of research and practice but first a few examples of utility functions.

1. Exponential
   The function
   $$U(x) = -e^{-\alpha x}$$
   where α is a strictly positive constant is quite often used. The fact that this utility takes negative values is irrelavant.

[15] Because there is only a *partial* order on functions rather than the *total* order that pertains to real numbers

2. Logarithmic
$$U(x) = log(x)$$
Note that we restrict x to be strictly positive. Observe that this function is "very steep" near to the number 0 so it separates outcomes with small positive values radically. Also, if an outcome may be expressed in the following way
$$A = 0I_E + XI\Omega \setminus E$$
where E has positive probability then
$$U(A) = log(0)I_E + log(X)I_{\Omega \setminus E}$$
and we would have to set $E(log(A)) = -\infty$. So that this utility penalises the possibility of a zero outcome very strongly.

3. Power
$$U(x) = ax^a$$
for some non-zero constant, $a \leq 1$.

4.1 Portfolio choice using Utility Functions

Consider the case of an investor with utility function U, initial wealth W and a choice from the n (basic) securities, $d_1, d_2, \dots, d_n$. If X denotes the (random) final wealth arising from holding the security d_k in the quantity θ_k for $1 \leq k \leq n$ then we may define this investors investment problem by

$$\begin{aligned} \text{maximise} \qquad & E(U(X)) \\ \text{subject to} \qquad & \sum_1^n \theta_k d_k = X \\ \text{and} \qquad & X > 0 \\ \sum_1^n \theta_k d_k(0) \quad & \leq W \end{aligned} \tag{14}$$

Here $d_k(0)$ denotes the price of the security d_k at time 0 whereas d_k is its random value at the end of the investment period. There is a theorem, which states that in the absence of arbitrage, if U tends to ∞ as $x \longrightarrow \infty$ and there is at least one portfolio (choice of the θ's) which makes X strictly positive, then this problem has a solution. Moreover, and this is not difficult to prove, one has the equality

$$\sum_1^n \theta_k^* d_k = W$$

when θ^* is the portfolio that solves the problem[16]. In this case one can analyse for the specific amounts θ_k as follows. The problem is to maximise

$$E(U(\sum_1^n \theta_k d_k))$$

subject to

$$\sum_1^n \theta_k^* d_k(0) = W.$$

One can use the method of Lagrange multipliers: Let

$$L = E(U(\sum_1^n \theta_k d_k) - \lambda(\sum_1^n \theta_k^* d_k - W).$$

In order to find the values of θ_k that maximise this function we must set each of the partial derivatives of L with respect to θ_k equal to zero. Letting $X^* = \sum_1^n \theta_k^* d_K$ it is easy to see that this gives the set of equations

$$E(U'(X^*)d_k) = \lambda d_k(0)$$

which along with the 'budget constraint'

$$\sum_1^n \theta_k^* d_k(0) = W$$

gives $n + 1$ equations in the $n + 1$ unknowns $d_1, d_2, \ldots, d_n, \lambda$. It turns out that these simple equations are quite important for Investment Theory. In particular, if there is a risk-free asset in our portfolio whose initial value is (or can be assumed to be) 1 and whose return for the period considered is R the new can solve one of our k equations above to get

$$\lambda = E(U'(X^*)R.$$

[16]If one had strict inequality one could add a fraction of θ^* to improve the solution, contradiction

Putting this value for λ back into the other equations gives

$$d_k(0) = \frac{E(U'(X^*)d_k)}{RE(U'(X^*))}.$$

So we see a relationship between the value of the optimal portfolio, the derivative of our utility function , the risk-free rate, the final (random) value of the $k-th$ security and the initial price of that security. It is in turning this relationship on its head and seeing it as a way of determining prices that leads to some interesting ideas. Let's look at a special case.

4.2 The Log Optimal Portfolio

Let us take as our utility function, $U(x) = log(x)$. The optimal portfolio for this case is called the log optimal portfolio. Suppose that our initial budget, W, is equal to 1. Let X^* denote the final wealth of the portfolio which maximises the expected logarithm of final wealth. Since $W = 1$ we can regard X^* as the *log optimal return*. From our equations above

$$E(U'(X^*)d_k) = E(\frac{d_k}{X^*}) = \lambda d_k(0).$$

Using linearity we see that

$$\sum_1^n E(\frac{\theta_k d_k}{X^*}) = \sum_1^n \lambda\theta_k d_k(0).$$

but since $E()$ is linear and the initial budget is 1 then this says that

$$E(\frac{X^*}{X^*}) = \lambda.$$

Or that $\lambda = 1$. As a consequence we get

$$d_k(0) = E(\frac{d_k}{X^*})$$

for each k. This gives us an expression for the price of each of our basic assets in terms of the expectation of its payoff and the log optimal return. Since the price of a linear combination of the basic assets is simply the corresponding linear combination of the prices, this formula allows us to price any portfolio made up from our basic assets.

4.3 Portfolio Growth

Consider the following general formulation of an investment situation. One starts (at time 0) with an amount, X_0, which is transformed into the random amount, $X_1 = R_1 X_0$, at time 1. Indeed at time k the wealth accrued is given by

$$X_k = R_k X_{k-1}.$$

Here the random variable, R_k, denotes the (random) return over the time period from $k-1$ to k, and X_{k-1} is the (random) wealth at time $k-1$. For our purposes we assume that the discrete time process, (R_k), consists of mutually independent and identically distributed random variables. It is easy to see that

$$X_k = R_k R_{k-1} R_{k-2} \ldots R_2 R_1 X_0.$$

for a general k (with suitable interpretation of the formula for small values of k). We can rewrite this last relation as

$$log X_k = log X_0 + \sum_{i=1}^{k} log R_i$$

and from this follows immediately

$$log(\frac{X_k}{X_0})^{\frac{1}{k}} = \frac{1}{k}\sum_{i=1}^{k} log R_k.$$

Now you may be wondering why, at this point, have I introduced a $k-th$ root? Bear with me! The right side of the last equation has been the subject of much attention from the mathematics community down the years. Should the random variables R_k be integrable the the *law of large numbers* states that

$$\frac{1}{k}\sum_{i=1}^{k} log R_k \longrightarrow E(log R_1)$$

almost surely, that is, the average of the R_k's converges to the common mean of the R_k's. Writing $m = E(log(R_1))$ and taking in to account our earlier equations we see that

$$X_k \longrightarrow X_0 e^{km}.$$

as $k \longrightarrow \infty$. This is stating that for a large number of time periods the wealth grows (approximately) exponentially with rate m. So if one is investing for the long term and one wants to maximise growth then one should seek the largest value of m that one can achieve. But since $m = E(log(R_1))$ this amounts to maximising the expectation of the logarithm of R_1. The reader will recognise that we are close to associating a Utility Function with this wealth process. Indeed, if we add $logX_0$ to m then we get $logX_1$ and maximising $E(logX_1)$ is equivalent to maximising $E(log(R_1))$. This reduces the multiperiod problem to a single period problem[17]. To maximise long term growth the investor must determine the (log) optimal strategy for the first time period and then apply this repeatedly.

4.4 Portfolios with several assets

We consider a portfolio consisting of a risk - free asset, S^0 [18], and $n \geq 1$ stocks, S^i, $1 \leq i \leq n$. The stocks will follow logarithmic Brownian Motions;

$$S^i(t) = S^i(0) + \int_0^t S^i(s)\mu_i ds + \int_0^t S^i(s)dZ^i(s)$$

here Z^i, for $i = 1, 2, \ldots, n$ is a Brownian Motion with $Cov(Z^i(t), Z^j(t)) = \sigma_{ij}t$ and the μ_i's are constants. The nxn matrix $[\sigma_{ij}]$ is assumed to be non-singular. If we form a portfolio, V, of these $n+1$ assets holding asset an amount, ψ_i of the asset i, until time t, then value of the portfolio at any time, t, is

$$V_t = \sum_{i=0}^{n} \psi_i S^i(t).$$

Now we can express the quantities ψ_i in a special way; Let

$$\alpha_i = \frac{\psi_i S^i(0)}{V_0}.$$

Then α_i is simply the proportion of the *initial value* of the portfolio invested in the asset S^i. We refer to these numbers as the *weights* of the assets, S^i, $0 \leq i \leq n$. In terms of these weights the value of the portfolio at any time t is

$$V_t = \sum_{i=0}^{n} \alpha_i \frac{V_0}{S^i(0)} S^i(t).$$

[17] As one might expect, because each of the R_k's is probabilistically speaking identical with all the others bar that they are independent.

[18] It has the usual dynamics

One can consider questions about how one should choose the numbers α_i in order to achieve a particular investment goal, indeed we will do this subsequently. One particular reason for writing our portfolio value in this form is that it gives a simple expression for the portfolio return over $[0,t]$. Indeed

$$\begin{aligned}
\frac{V_t - V_0}{V_0} &= \frac{\sum_{i=0}^{n} \alpha_i \frac{V_0}{S^i(0)} S^i(t) - V_0}{V_0} \\
&= \sum_{i=0}^{n} \alpha_i \frac{1}{S^i(0)} S^i(t) - 1 \\
&= \sum_{i=0}^{n} \alpha_i \left(\frac{S^i(t) - S^i(0)}{S^i(0)}\right)
\end{aligned}$$

So the return is the weighted sum of the returns from the individual assets. Note that this is only true because of the specific way in which we have defined the weights, α_i. In general, a convex combination of assets has a return which is different from the convex combination of the returns [19].

At this point the argument takes an unfortunate twist. It is argued, incorrectly, that the process V is lognormal[20]. From this it is argued that the expected logarithm of $\frac{V_t}{V_0}$ has the form

$$\sum_{i=1}^{n} \alpha_i \mu_i t - \frac{1}{2} \sum_{i,j}^{n} \alpha_i \alpha_j \sigma_{ij} t$$

while the variance of the same term is

$$\sum_{ij} \alpha_i \alpha_j \sigma_{ij} t$$

These two terms give a convenient form for the growth rate and variance of the portfolio [21]. One can now consider the problem of choosing the weights, α_i, to maximise growth subject to constraints (perhaps). We shall leave this topic now. I hope to produce a rigorous argument along the lines traced out above and to use the log optimal portfolio to give a derivation of the Black-Scholes formula. I expect this to be finished in the autumn of 1999. Please, please, inform me of typos', ommissions, errors, bad type setting and so on[22]. I am including an annex on numeraires as preparation for the next sections. Some of you will have seen this already.

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[19] Try the functions, x and x^2 over the interval $[1,2]$ in the proportions $\frac{1}{4}$ and $\frac{3}{4}$

[20] See Luenberger's book, Investment Science, OUP 1997, pp 428

[21] Convenient but incorrect, perhaps an approximation is intended here...

[22] My e-mail address is on the title page

The Numeraire

We suppose that the 'usual conditions' prevail in so far as the filtration and probability measure are concerned. In our model the numeraire, S, shall be supposed to be a measurable adapted process such that

$$E(\int_0^T |S(t,\omega)|dt) < \infty$$

and to satisfy the stochastic equation,

$$S_t = S_0 + \int_0^t r(s)S_s ds$$

where is $r(s)$ is a strictly positive measurable adapted process on $[0,T] \times \Omega$. We will assume further that r is a bounded continuous process and remark here that this means that r is a predictable process. We will also assume that $S_0 = 1$, one can achieve this by a scaling of all assets in any case. The first condition satisfied by S entails that

$$\int_0^T |S(t,\omega)|dt < \infty$$

for P-almost every $\omega \in \Omega$. The question of whether or not such a process exists will be dealt with later, for now we observe that any process satisfying the conditions above will be P-almost surely continuous because,

$$|S_t - S_s| = |\int_s^t r(k)S_k dk| \leq \int_s^t |S_k r(k)|dk \leq M\int_s^t |S_k|dk$$

where M is a bound for the process $r(t)$. The result now follows because $k \longmapsto |S_k(\omega)|$ is an integrable function for P-almost every $\omega \in \Omega$. We now know that, P-almost surely, the expression

$$\int_0^t r(s)S_s ds$$

is differentiable with derivative, $r(t)S_t$. Accordingly it satisfies, P-almost surely the differential equation

$$\frac{dS_t}{dt} = r(t)S_t$$

with the boundary condition, $S_0 = 1$. This first order linear equation may be solved by means of an integrating factor to yeild

$$S_t = e^{\int_0^t r(s)ds},$$

for P-almost every $\omega \in \Omega$. This result applies equally well to any other security whose price process has the same form as S. Looking at the process,

$$e^{\int_0^t r(s)ds},$$

we note that it certainly satisfies the assumptions made about S. Moreover, any process satisfying those assumtions will be equal to this process P-almost surely. Given that we identify processes whose paths coincide off of a P null set, we see that our solution is unique.

The Market and Riskless Assets

We assume that our market is *arbitrage free*. What does this mean mathematically? It amounts to this; an arbitrage opportunity is an admissible self financing portfolio with value at time t, V_t, but in particular, $V_0 = 0$ and $E(V_T) > 0$. For an elaboration of the terminology see Harrison and Pliska. We will define an asset to be riskless if it satisfies equations similar to that for S. Indeed Y is riskless iff

$$E(\int_0^T |Y_s| ds) < \infty$$

and

$$Y_t = Y_0 + \int_0^t k(s) Y_s ds$$

for some bounded strictly positive adapted continuous process, $k(s)$. We note that this entails that $k(s)$ is measurable (the 'usual conditions' help here). What we aim to prove here is this;

Theorem 1 *If the market has no arbitrage then for P-almost every $\omega \in \Omega$, $k(s, \omega) = r(s, \omega)$ for λ- almost every $s \in [0, T]$.*

Proof 2 *Let's begin by considering an extreme case of this result; when both r and k are constants one easily solves the respective equations for S and Y to obtain, in the case of S,*

$$S_t = e^{rt}.$$

With a similar result for Y. Now if, say, $k > r$ then by borrowing Y_0 units of S at time 0, purchasing Y, and waiting until, say, time T. Then our assets are Y_T and our liabilities $Y_0 e^{rT}$. Now

$$Y_T = Y_0 e^{kT} > Y_0 e^{rT}.$$

So our assets are strictly greater than our liabilities and all for a zero start up cost. This is an arbitrage opportunity, so we cannot have $k > r$. If the reverse inequality prevails then one sells short Y and invests in S for a positive time period. Once again an arbitrage opportunity appears. So all that we are left with is $k = r$. In the general case, we cannot argue in this simple manner. Even if k and r are deterministic functions, proving that $k(s) > r(s)$ on $[0, T]$ is false *does not establish the reverse inequality because function f can majorise function g for some points but the reverse can happen at other points* [23]*. To begin; we observe that the predictable σ-field is generated by the collection of sets* [24]

$$(s, t] \otimes B \quad \textit{for} \quad 0 \leq s < t \leq T \textit{ and } B \in \mathcal{F}_s$$

and

$$\{0\} \otimes A \textit{ for } A \in \mathcal{F}_0.$$

Consider a set of the (first) form, $(s, t] \otimes B$. We construct a self-financing trading strategy; for $\omega \in \Omega \setminus B$ we hold no assets over $[0, T]$. For $\omega \in B$, at time s we borrow $Y_s(\omega)$ units of S and purchase Y. At time t, for $\omega \in B$, we realise our asset, $Y_t(\omega)$, and meet our liability $Y_s(\omega) exp(\int_s^t r(s) ds$. Now,

$$Y_t(\omega) I_B = Y_s(\omega) e^{\int_s^t k(l) dl} I_B$$

while the difference of our assets and liabilities is

$$Y_s(\omega) I_B (e^{\int_s^t k(l) dl} - e^{\int_s^t r(l) dl}).$$

We could invest this portfolio in the asset S until time T. Since $Y_s(\omega)$ is strictly positive, should B have positive measure and it were true that $k(s) > r(s)$, Lebesgue almost everywhere over $(s, t]$ on the set B, except possibly for a null set of points in B , then we would have created an arbitrage

[23]Which is a long way of saying that the ordering on functions is not a *total* order
[24]I use the symbol $\otimes$ to denote cartesian product

opportunity because the time T value of the expectation of this portfolio would be strictly positive. By modifying our portfolio; selling Y short at time s, investing in S on B until time t. The assumption that $r(s) > k(s)$ Lebesgue almost everywhere over $(s, t]$ on the set B (except possibly for a null set of points in B) gives rise to another arbitrage opportunity. To conclude; for every interval, $(s, t] \subseteq [0, T]$, and every set, including null sets *and sets of positive probability, $B \in \mathcal{F}_s$, we cannot have $r > k$ or $k > r$ Lebesgue almost everywhere over $(s, t]$ on B, except, possibly, for a null set of points in B. Suppose now that $A \in \mathcal{F}_0$ and has positive probability. Can we have $r(0) > k(0)$ for almost every $\omega \in A$? Or vice-versa? Consider the following argument, let $k(0) > r(0)$ and τ be the (stopping) time which for $\omega \in A$ is*

$$\tau(\omega) = inf\{t \in (0, T] : k(t, \omega) = r(t, \omega)\},$$

and is otherwise zero. If the set on the right side above is empty we define $\tau(\omega)$ to be T. Since $k - r$ is a predictable process, the debut of the Borel set $\{0\}$ under $k - r$ is a stopping time (and τ is built out of this time and the set A). From these facts it is not difficult to show that τ is indeed a stopping time. Since $k - r$ is continuous and by hypothesis strictly positive on A then τ is strictly positive on A. We define a self financing trading strategy by stipulating that on $\Omega \setminus A$ we hold no assets over $[0, T]$, while on A we borrow against S to buy Y and hold until τ. Our assets are then

$$Y_0 e^{\int_0^{\tau(\omega)} k(s,\omega)ds}$$

while our liabilities are

$$Y_0 e^{\int_0^{\tau(\omega)} r(s,\omega)ds}$$

The difference is strictly positive P-almost surely and can be invested until time T (the time at which we formally decide if an arbitrage opportunity has occured). Formally, if ϕ denotes the holding in S and θ the holding in Y then

$$\phi_t(\omega) = (Y_0 \int_0^{\tau} (k(s) - r(s))ds) I_{A \cap \{\tau \leq t\}} - I_{A \cap \{\tau > t\}}$$

while

$$\theta_t(\omega) = I_{A \cap \{\tau > t\}}.$$

Of course this portfolio, which had zero start up cost, and comprises an admissible trading strategy, is an arbitrage opportunity. If the inequality between

k and r is reversed then a short selling argument similar to that employed above generates another arbitrage opportunity. So, for every *set, A, in $\mathcal{F}_0$ we cannot have $k > r$ or $r > k$ on A except, possibly, on a null set in A for . What we have shown here, so far, is that the collection of subsets, $\mathcal{M}$, of $[0, T] \otimes \Omega$ for which we cannot have $k > r$ or $r > k$ Lebesgue almost everywhere in time except, possibly, on a P null set in Ω, contains the generators of the predictable σ-field. We now consider the collection $\mathcal{M}$.*

1. *The elements of $\mathcal{M}$ are closed under finite intersections.*
2. *If E and F belong to $\mathcal{M}$ and $E \subseteq F$ then $F \setminus E$ is clearly in $\mathcal{M}$.*
3. *If (E_n) is an increasing sequence of sets in $\mathcal{M}$ then $\bigcup E_n$ is in $\mathcal{M}$ because the union of a countable number of null sets is a null set.*
4. *The set $[0, T] \otimes \Omega$ lies in $\mathcal{M}$ by an argument identical to one employed for $(s, t] \otimes B$ above.*

It follows from the monotone class theorem that $\mathcal{M}$ contains the predictable σ-field. If we now consider the predictable set*, $\{(t, \omega) : k(t, \omega) > r(t, \omega)\}$, then k can exceed r (Lebesgue almost everywhere) over time on this set only on an Ω set of probability zero and the reverse inequality* cannot *hold on this set. A similar remark applies to the set defined by the reverse inequality. So, we must conclude that for P-almost every $\omega \in \Omega$ we must have $r(t, \omega) = k(t, \omega)$ for Lebesgue almost every $t \in [0, T]$.*

5 Aliter: Ian's Proof

At the stage where we are considering a set of the form $E \otimes (s, t]$, $E \in \mathcal{F}_s$, Ian says we should argue as follows. Consider the continuous process, $k - r$