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Multipolar vortices and algebraic curves

By Darren Crowdy

Department of Mathematics, Imperial College of Science, Technology and Medicine, 180 Queen's Gate, London SW7 2BZ, UK (d.crowdy@ic.ac.uk)

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This paper demonstrates that analytical solutions to the steady two-dimensional Euler equations possessing all the qualitative properties of multipolar vortices observed experimentally and numerically can be constructed using the theory of algebraic curves and quadrature domains. The solutions consist of a finite set of line vortices superposed on finite-area patches of uniform vorticity. By way of example, new solutions are presented in which the support of the vorticity is a quintuply connected vortex patch with five vorticity maxima modelling a symmetric pentapolar vortex consisting of a central core vortex, four satellite vortices and four enclosed zones of irrotational fluid between the core and the satellite vortices. The method of construction is amenable to the derivation of exact solutions of the two-dimensional Euler equations having vorticity distributions of even greater geometrical complexity.

Keywords: multipolar vortices; quadrature domains; algebraic curves

1. Introduction

Coherent vortical structures are now known to constitute an important feature of many two-dimensional and quasi-geostrophic flows (McWilliams 1984). The Rankine vortex is a classical exact solution for a uniform circular patch of vorticity (Saffman 1992) and represents one of the simplest examples of a solution of the two-dimensional Euler equations with distributed vorticity. Maxwell (1861*a*,*b*) used the Rankine vortex model in his early attempts to describe the nature of electrical and magnetic phenomena. He referred to these vortex structures as 'molecular vortices'. In this paper, we follow in the spirit of these early investigations but base our approach on a different 'molecular vortex'. It will be shown that it is possible to 'nonlinearly superpose' (in a manner to be explained) a variant of the usual Rankine vortex solution, referred to herein as the *shielded Rankine vortex*, in such a way as to construct compound vortical equilibria of the two-dimensional Euler equation consisting of much more complicated regions of distributed vorticity. The resulting solutions fall within the class of solutions of the Euler equations which have become collectively known as *multipolar vortices*.

Coherent vortices arise in many aspects of astrophysical, geophysical and meteorological fluid dynamics. In an initial state of randomly distributed vorticity, for example, the cascade of energy to larger scales is responsible for the formation of such vortical structures (Legras *et al.* 1988). While monopoles and dipoles are the most ubiquitous structures (characterized by one and two vorticity maxima, respectively), laboratory experiments (Orlandi & Van Heijst 1992; Van Heijst *et al.* 1989, 1991) and numerical simulations (Carton & Legras 1992; Carton *et al.* 1989; Polvani & Carton 1990) have shown that higher-order structures, such as tripoles and quadrupoles (Beckers & van Heijst 1998; Carnevale & Kloosterziel 1994; Morel & Carton 1994) (as well as even higher-order structures) arise from the instability of isolated circular vortices, usually with zero total circulation. A tripole is characterized by three vorticity maxima: a central core region with vorticity of one sign surrounded by two satellite vortices both of opposite sign. A quadrupole has a central core surrounded by a triangular array of three such satellites. The class of vortices of this general kind has been dubbed *multipolar* and the formation, structure and stability properties of such vortices are a topic of much recent research activity. Typically, such vortex structures rotate at a constant angular velocity. The reader is referred to Carnevale & Kloosterziel (1994) for a detailed discussion of the properties of multipolar vortices.

Owing to the complicated structure of these multipolar vortices, most investigations of them have involved either laboratory experiments or full numerical simulations, although simple point vortex models can often capture many aspects of realistic flow situations (see, for example, Velasco Fuentes *et al.* 1996). It is of interest, from a theoretical point of view, to find effective models of such vortices, or ideally, some mathematical solutions of the Euler equations which resemble multipolar vortices and which can be studied explicitly. In this vein, Kloosterziel & Carnevale (1999) examined the possibility of approximating the evolutionary dynamics between these equilibria by low-order dynamical systems while, with a view to gaining an understanding of these structures as mathematical solutions of the Euler equations, the present author recently pointed out (Crowdy 1999) that there exists a class of exact solutions to the steady two-dimensional Euler equations which share all the qualitative properties of multipolar vortices observed in practice. The solutions in Crowdy (1999) are *finite-area* patches of non-zero vorticity and so, unlike simple point-vortex models, provide insight into the *shapes* of multipolar vortical equilibria of the Euler equation. Mathematically, the solutions are non-trivial generalizations of the geometrically trivial *shielded Rankine vortex* (defined in § 2).

The class of solutions derived in Crowdy (1999) has the property of being 'invisible' in that the irrotational velocity field induced outside the support of the vorticity is identically zero. Any two such vortices therefore only interact when they overlap. This prompts a very natural question: what happens when such vortices *do* overlap? In particular, can the overlapping of such vortices produce more coherent vortex equilibria of the two-dimensional Euler equations, perhaps with vortical regions having even more complex shapes?

Crowdy (2001*a*) has answered this question in the affirmative and has shown by explicit construction that such compound solutions are possible when a collection of three or more *shielded Rankine vortices* merge in an annular configuration forming a *doubly connected* vorticity region enclosing a central region of irrotational fluid. This result raises the intriguing possibility of constructing solutions of the steady Euler equation consisting of vortical regions with even more complex topologies. This would be interesting from a mathematical point of view in that it might lead to a general 'superposition principle' for vortex patches. Physically, it might also lead to the construction of more realistic multipolar solutions of the Euler equation. In particular, multipolar vortices typically display regions of irrotational fluid between the core and satellite vortices; the solutions of Crowdy (1999) do not have this feature.

2. The shielded Rankine vortex

A classical exact solution of the steady two-dimensional Euler equations governing the motion of an ideal fluid is given by the velocity field $\boldsymbol{u} = (0, V(r))$ (in plane polar coordinates (r, θ)) where

$$V(r) = \begin{cases} \dfrac{\omega_0 r}{2}, & r \leqslant r_0, \\ \dfrac{\omega_0 r_0^2}{2r}, & r > r_0, \end{cases} \tag{2.1}$$

depending on the parameter pair (ω_0, r_0), where ω_0 is the vorticity, while r_0 is a geometrical parameter giving a measure of its size. This simple exact solution is known as the Rankine vortex (Saffman 1992). Now suppose that a line vortex with circulation equal and opposite to the total circulation of the Rankine vortex is placed at its centre. The associated velocity field is then given by $\boldsymbol{u} = (0, \hat{V}(r))$, where

$$\hat{V}(r) = \begin{cases} \dfrac{\omega_0 r}{2} - \dfrac{\omega_0 r_0^2}{2r}, & r \leqslant r_0, \\ 0, & r > r_0. \end{cases} \tag{2.2}$$

The flow exterior to the patch of vorticity now vanishes identically and the total circulation of the combined vortical structure is zero. Because of this, the solution (2.2) will be referred to as a *shielded Rankine vortex*.

The flow (2.2) is incompressible, so there is an associated stream function which is related to the radial and tangential components of a general velocity field (U, V) (in plane polar coordinates) by the formula

$$2\mathrm{i}\psi_z = \mathrm{e}^{-\mathrm{i}\theta}(U - \mathrm{i}V), \tag{2.3}$$

where $z = x + \mathrm{i}y$ and (x, y) denote plane Cartesian coordinates. By integration, the stream function associated with the particular solution (2.2) can be seen to be

$$\psi(z, \bar{z}) = -\tfrac{1}{4}\omega_0 \left(z\bar{z} - \int^z S(z')\,\mathrm{d}z' - \int^{\bar{z}} \bar{S}(z')\,\mathrm{d}z' \right), \tag{2.4}$$

to within an inconsequential real constant, where

$$S(z) \equiv r_0^2/z. \tag{2.5}$$

The function $S(z)$ happens to be the *Schwarz function* (Davis 1974) of the circle $r = r_0$, and this observation is important in providing a route to generalization of the monopolar shielded Rankine vortex to higher-order multipolar structures with non-trivial geometry (see Crowdy 1999).

It will be shown that the shielded Rankine vortex solution can be used as a 'building-block' solution in the construction of higher-order structures. Due to the fact that there is no induced flow outside the support of the vorticity then, provided they do not overlap, an array of such solutions, perhaps with varying parameter pairs (ω_0, r_0), can be arbitrarily superposed in the plane to form other global equilibria of the Euler equations. For example, a global equilibrium constructed in this way is depicted in figure 1, where five different shielded Rankine vortices are arbitrarily superposed in the plane. This equilibrium has a geometrically trivial distribution of

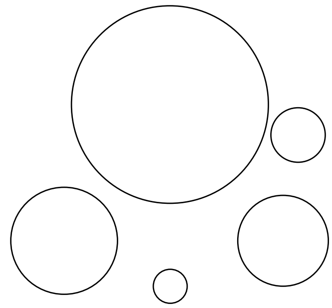

Figure 1. An array of five non-overlapping shielded Rankine vortices. There is a line vortex (not shown) at the centre of each disc.

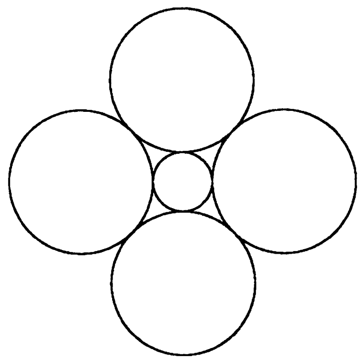

Figure 2. Geometrical arrangement of five touching shielded Rankine vortices. There is a line vortex (not shown) at the centre of each disc.

vorticity consisting of an assortment of disconnected circular patches each with a superposed central line vortex.

In contrast to the illustrative situation in figure 1, where five shielded Rankine vortices are *arbitrarily* superposed in the plane such that they do not overlap or touch, in figure 2 five shielded Rankine vortices are now carefully positioned in the plane in such a way that they are just touching. The question to be addressed here is whether this equilibrium can be 'continued' to form a more complicated compound equilibrium. This continuation procedure, and how to perform it, is the principal subject of this paper. We will show, by explicit construction, that a geometrically more complicated pentapolar equilibrium can be constructed by the 'merging' of the five touching shielded Rankine vortices shown in figure 2.

In Crowdy (2001*a*) the theory of Schwarz functions (Davis 1974) was combined with conformal mapping theory to construct vortical structures having *doubly connected* vortical regions. In this paper, the mathematical approach of Crowdy (2001*a*) is generalized in a non-trivial way by exploiting the theory of algebraic curves and quadrature domains to produce a model of a pentapolar vortex having a *quintuply connected* vortical region with four enclosed islands of irrotational fluid. In presenting this case study, we hope to illustrate the potential of the new approach as a method for the construction of multipolar equilibria of the two-dimensional Euler equations having vortical regions with very complex topological structure.

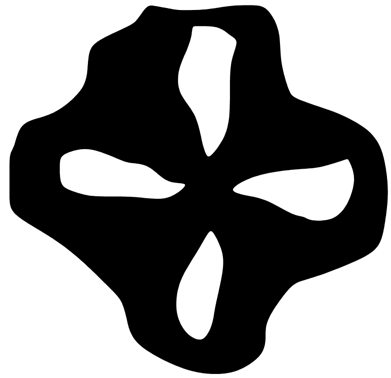

Figure 3. A quintuply connected patch of vorticity.

3. Mathematical formulation

Let the domain D be a non-empty, open, connected, bounded subset of the complex plane $\mathbb{C}$. Let the connectivity of the domain D be N. See figure 3 where a typical domain with connectivity $N = 5$ is shown. Let ∂D denote the boundary of D and $\bar{D}$ the closure of D. We will seek solutions of the two-dimensional Euler equations consisting of finite-area patches of vorticity which is uniform except for a finite set of line vortex singularities superposed on the patch. The set D just described will constitute the region of non-zero uniform vorticity. The $N - 1$ enclosed regions will be taken to be irrotational (i.e. no vorticity).

Motivated by the form of the stream function given in (2.4), define a stream function $\psi(z, \bar{z})$ as follows:

$$\psi(z,\bar{z}) = \begin{cases} z\bar{z} - \int^{z} S(z')\,\mathrm{d}z' - \int^{\bar{z}} \bar{S}(z')\,\mathrm{d}z', & z \in \bar{D}, \\ 0, & z \in \mathbb{C}\backslash\bar{D}, \end{cases} \tag{3.1}$$

where $S(z)$ is a function to be specified. It is clear from (3.1) that the flow is quiescent exterior to the vortex patch (and therefore trivially irrotational). The associated vorticity inside the patch D takes the uniform value -4 except at any singularities of the function $S(z)$. This is an arbitrary choice made for convenience. So far, we have made no specifications on the domain D or the function $S(z)$.

Now suppose that it is possible to construct a function $S(z)$ and a domain D such that the following conditions are satisfied.

(a) $S(z)$ is meromorphic in D with poles at the finite set of points $\{z_j\}$ ($z_j \in D$).

(b) $S(z)$ is continuous on $\bar{D}\backslash\{z_j\}$.

(c) $S(z)$ satisfies $S(z) = \bar{z}$ for $z \in \partial D$.

(d) $S(z)$ has only simple pole singularities in D, with real residues.

(e) Near each element of the set $\{z_j\}$, $S(z)$ is of the form

$$S(z) = \frac{\mathit{\Gamma}_j}{4\pi(z - z_j)} + \bar{z}_j + \mathcal{O}(|z - z_j|), \tag{3.2}$$

where $\mathit{\Gamma}_j \in \mathbb{R}$.

If (3.1) is to represent a steady solution of the two-dimensional Euler equation, there are a number of physical constraints that must be satisfied. Henceforth, we refer to D as the vortex patch. First, there is a kinematic requirement that the boundaries of the patch are streamlines. Second, a dynamical requirement is that the fluid pressure is continuous across the vortex jumps on the boundaries of the patch and this is well known to be equivalent to continuity of fluid velocity (Saffman 1992).

Let (u, v) denote the Cartesian components of the velocity field $\boldsymbol{u}$ with respect to Cartesian coordinates (x, y). It is straightforward to show that (u, v) follow from the stream function ψ by means of the formula

$$u - \mathrm{i}v = 2\mathrm{i}\psi_z. \tag{3.3}$$

Using (3.1),

$$u - \mathrm{i}v = 2\mathrm{i}\psi_z = 2\mathrm{i}(\bar{z} - S(z)). \tag{3.4}$$

The fluid velocity exterior to the patch (in $\mathbb{C}\backslash\bar{D}$) is zero, so the dynamical requirement that velocities are continuous on the boundary of the patch dictates that

$$\bar{z} - S(z) \quad \text{on } \partial D. \tag{3.5}$$

However, this is satisfied by condition (c) above. Moreover,

$$\begin{aligned} \mathrm{d}\psi &= \psi_z \, \mathrm{d}z + \psi_{\bar{z}} \, \mathrm{d}\bar{z}, \\ &= (\bar{z} - S(z)) \, \mathrm{d}z + (z - \bar{S}(\bar{z})) \, \mathrm{d}\bar{z}. \end{aligned} \tag{3.6}$$

A streamline is a curve on which ψ is constant, so $d\psi = 0$ on a streamline. This is true on ∂D, again by condition (c) above. Therefore, both the kinematic and dynamic boundary conditions on the boundary of the patch are satisfied by (3.1).

By condition (a) above, the vorticity associated with (3.1) is uniform inside the patch except for the vortical singularities associated with the finite set of singularities $\{z_j\}$ of the function $S(z)$. By condition (d) above, all such singularities are actually *line vortices*. Moreover, by condition (e) above, near each z_j the local velocity field has the form

$$u - \mathrm{i}v = -\frac{\mathrm{i}\Gamma_j}{2\pi(z - z_j)} + \mathcal{O}(|z - z_j|). \tag{3.7}$$

The real number Γ_j therefore represents the strength (or circulation) of the line vortex at position z_j. Note that there is no constant term in the local expansion (3.7). This condition is crucial and corresponds to the fact that the line vortex is steady under the effects of the non-self-induced velocity field. This is a necessary physical condition required by the Helmholtz laws of vortex motion (Saffman 1992). Only if this condition is satisfied will (3.1) represent a consistent steady solution of the Euler equation.

It remains to ascertain whether a set D and a function $S(z)$ satisfying conditions (a)–(e) can be found. Crowdy has shown, by explicit construction using conformal mapping theory, that such solutions are possible in the simply connected case $N = 1$ (Crowdy 1999) and in the doubly connected case $N = 2$ (Crowdy 2001*a*). In what follows, we deliberately avoid the use of conformal maps and instead use the theory of quadrature domains and algebraic curves. By way of example, we show, by explicit construction, that such solutions also exist for $N = 5$. It will be clear that the present methods are, in principle, general enough to allow the possibility of constructing solutions with vortical regions of any finite connectivity.

4. Quadrature domains

The domains D and functions $S(z)$ required to satisfy the conditions stipulated in the previous section can be constructed by considering the theory of quadrature domains. The mathematical theory of quadrature domains is well developed (Aharonov & Shapiro 1976; Gustafsson 1983, 1988; Sakai 1982). The simplest example of a quadrature domain is a circular disc. For a circular disc D, of radius r, centred at the origin, the well-known mean value formula (Ablovitz & Fokas 1997) states that

$$\iint_D h(z)\,\mathrm{d}x\,\mathrm{d}y = \pi r^2 h(0), \tag{4.1}$$

where $h(z)$ is an arbitrary integrable (with respect to area measure) analytic function in the disc.

As a generalization of this case, we call a domain D a *quadrature domain* if the following *quadrature identity* holds for all $h(z)$ analytic in D and integrable over D with respect to area measure:

$$\iint_D h(z)\,\mathrm{d}x\,\mathrm{d}y = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} c_{kj} h^{(j)}(z_k), \tag{4.2}$$

for some set of (complex) coefficients $\{c_{kj}\}$ and some point set $\{z_k \mid k = 1, \ldots, m\}$ consisting of $m \geqslant 1$ distinct values, where m is a positive integer. The set $\{n_k \mid k = 1, \ldots, m\}$ is a set of positive integers and $\sum_{k=1}^{m} n_k$ is known as the *order* of the quadrature identity. Here $h^{(j)}(z)$ denotes the jth derivative of $h(z)$. The sets of complex numbers $\{c_{kj}\}$ and $\{z_k\}$ appearing in the right-hand side of (4.2) are known as the *quadrature data* of D. In the present application, the part of the quadrature data called $\{z_k\}$ will be exactly the line vortex positions mentioned in § 3, which is why the same notation as in the previous section has been employed.

We now summarize the mathematical results to be used in the construction of the vortex patch solutions. Aharonov & Shapiro (1976) have shown that D is a quadrature domain if and only if there exists a meromorphic function $H(z)$ in D such that

$$H(z) = \bar{z} \quad \text{on } \partial D. \tag{4.3}$$

Moreover, if the quadrature identity associated with the quadrature domain is given by (4.2) then the singularities of $H(z)$ are given by the set of points $\{z_k\}$. It can be seen that the function $H(z)$ is exactly the kind of function required in the construction of vortex patch equilibria as described in § 3 and, once a few additional constraints have been imposed upon it, the function $H(z)$ of Aharonov & Shapiro will correspond precisely to the function $S(z)$ required in the stream function (3.1) to represent an exact solution of the Euler equation. The set of points $\{z_k\}$ are the line vortex positions mentioned in the requirements for $S(z)$. Moreover, the vortex patches which satisfy all the requirements described in § 3 are now understood to be quadrature domains. If the patches contain enclosed regions of quiescent fluid then these quadrature domains will be multiply connected.

In the present application, the integer m in (4.2) will be the *number* of line vortices superposed on the patch. Because the singularities are all supposed to be line vortex singularities then $n_k = 1$ for all $k = 1, \ldots, m$. Finally, as will be shown in § 7 *b*, the values of c_{k0} will be directly related to the *strength* of the line vortex at z_k. In summary, specifying the positions and strengths of the line vortices corresponds to specifying the quadrature data of the quadrature domain, this domain being precisely the vortex patch on which the line vortices are superposed.

In this way, the problem of constructing multipolar equilibria of the Euler equations has been reduced to the construction of multiply connected quadrature domains satisfying certain supplementary conditions. Note that the *existence* of multiply connected quadrature domains has been established by Gustafsson (1983) using Riemann surface theory. To the best of the author's knowledge, however, no examples of explicit constructions of non-trivial quadrature domains of connectivity greater than two currently exist in the literature. In this paper, we devise a method to carry out such a construction by exploiting another important mathematical result of Aharonov & Shapiro; it is shown in Aharonov & Shapiro (1976) that if D is a quadrature domain then ∂D is part of an algebraic curve. More details of this important result are presented in § 5.

The method that will be presented represents a scheme for tackling the general mathematical problem of constructing multiply connected quadrature domains. The method is very general and flexible. Once the appropriate class of quadrature domains has been constructed, in the present application there is the additional complication of finding quadrature domains satisfying some additional physical constraints imposed by the Euler equation. This is done in § 7 *b*.

5. Algebraic curves

We now give a more detailed description of how to construct the relevant quadrature domains. It is known (Aharonov & Shapiro 1976) that, if D is a quadrature domain satisfying some identity (4.2), then its boundary ∂D is an *algebraic curve* given, to within a finite set of 'special points', denoted V_0, by

$$\partial D = \{z \in \mathbb{C} \mid \mathcal{P}(z, \bar{z}) = 0\} \backslash V_0, \tag{5.1}$$

where

$$\mathcal{P}(z, w) = \sum_{k,j=0}^{n} a_{kj} z^k w^j, \tag{5.2}$$

where $n = \sum_{k=1}^{m} n_k$ denotes the *order* of the quadrature identity (4.2) and where the coefficients $\{a_{kj}\}$ satisfy

$$a_{kj} = \bar{a}_{jk}. \tag{5.3}$$

The set of coefficients $\{a_{kj}\}$ will more conveniently be thought of as constituting the elements of a Hermitian matrix $\boldsymbol{A}$, where

$$\boldsymbol{A}_{kj} = a_{kj}. \tag{5.4}$$

The finite set V_0 will be important in the construction of the boundaries of the quadrature domains and will be discussed in more detail later. There is a normalization degree of freedom in the specification of the algebraic curve. This is fixed by specifying

$$a_{nn} = 1. \tag{5.5}$$

An alternative way of writing (5.2) is in the form

$$\mathcal{P}(z,w) = \sum_{j=0}^{n} w^j p_j(z), \tag{5.6}$$

where each $p_j(z)$ is a polynomial (in z) of degree at most equal to n, the order of the quadrature identity.

Of course, one expects that there must be some connection between the quadrature data, $\{c_{kj}\}$ and $\{z_k\}$, defining the quadrature identity (4.2) and the set of coefficients $\{a_{kj}\}$ defining the associated algebraic curve. Indeed, there is a *partial* connection embodied in the following theorem of Gustafsson (1983).

Theorem 5.1. *For a quadrature domain of order n satisfying the quadrature identity (4.2), the identity*

$$\frac{1}{\pi}\sum_{k=1}^{m}\sum_{j=0}^{n_k-1}\frac{j!c_{kj}}{(z-z_k)^{j+1}} \equiv a_{nn-1} - \frac{p_{n-1}(z)}{p_n(z)}, \tag{5.7}$$

where

$$p_{n-1}(z) = a_{nn-1}z^n + a_{n-1n-1}z^{n-1} + \cdots + a_{0n-1}, \tag{5.8}$$

$$p_n(z) = z^n + a_{n-1n}z^{n-1} + \cdots + a_{0n}, \tag{5.9}$$

sets up a one-to-one correspondence between the set of coefficients $\{c_{kj}\}$, $\{z_k\}$ and the last two columns (and rows, by the Hermitian property of $\boldsymbol{A}$) of the coefficient matrix $\boldsymbol{A}$.

It is important to realize that this connection between the coefficients is only partial; as emphasized by Gustafsson (1983), it is generally a difficult matter to determine the entire set of coefficients $\{a_{kj}\}$ (defining the algebraic curve) purely from knowledge of the quadrature data in (4.2).

As a general rule, one expects that the unknown coefficients defining the algebraic curve to be determined to within a number of degrees of freedom equal to the number of irrotational regions contained within the finite vortex patch. One way of understanding these degrees of freedom is to think of them as determining the area of each of the enclosed irrotational regions; to any given quadrature identity, there correspond many different quadrature domains, each having enclosed irrotational regions of different areas.

These degrees of freedom in defining the algebraic curve bounding the quadrature domain turn out to be crucial when seeking solutions of the steady Euler equations. Not every quadrature domain is a solution of the steady Euler equations. Indeed, most are not. Rather, only those quadrature domains which are consistent with the physical requirement dictated by the Helmholtz law that the line vortices are stationary under the effects of the non-self-induced terms in the local velocity field are physically admissible solutions. In order to impose this additional requirement, it will be necessary to use up some of the remaining degrees of freedom in defining the algebraic curve. Physically, the condition of stationarity of the line vortices will be seen to set the size/area of the enclosed irrotational regions.

6. Special points of quadrature domains

Gustafsson (1988) has also discussed how points of contact of disconnected circular domains leave so-called 'special points' z_s inside the domain as the domain changes so as to become connected. Such special points are defined as being *isolated* solutions of

$$\mathcal{P}(z_s, \bar{z}_s) = 0, \tag{6.1}$$

and do not, in general, constitute part of the continuous boundary ∂D of the quadrature domain D. These points constitute the finite set V_0 referred to in § 6. Moreover, at such points, it is known that the following holds:

$$\frac{\partial \mathcal{P}(z_s, \bar{z}_s)}{\partial z} = \frac{\partial \mathcal{P}(z_s, \bar{z}_s)}{\partial \bar{z}} = 0. \tag{6.2}$$

We refer the reader to Gustafsson (1988) for a detailed discussion of these special points. The important result from the work of Gustafsson (1988) that will be used in the sequel is that, for connected quadrature domains formed by two touching circular quadrature domains 'merging' together (in the present physical application, this corresponds to the merging of two shielded Rankine vortices), the point of contact between the initially touching discs leaves a 'special point' in the interior of the resulting connected quadrature domain. By extension, for complicated domains formed by the merging of *multiple* touching discs, for every initial point of contact between touching circles, one expects a 'special point' to appear inside the resulting connected domain. For quadrature identities possessing certain geometrical symmetries, it is natural to seek associated quadrature domains sharing the same symmetries. In certain cases, the symmetries of the quadrature identity and associated domain can be used to deduce information on where such special points should be. This is precisely what is found, using ideas from algebraic geometry, in Gustafsson (1988). It is sometimes possible to exploit these symmetries as well as the above-mentioned properties concerning the special points of the domain to deduce important information about the 'unknown' data of the matrix $\boldsymbol{A}$. This will be done in the context of a specific example in § 7.

The mathematically important notion of 'special points' on quadrature domains has been introduced in this section because of their importance in actually constructing quadrature domains (see the next section). It will turn out that the set V_0 also has an important *physical* significance.

7. An exact pentapolar vortex

In Crowdy (2001*a*), using conformal mapping theory, a multipolar vortex was constructed by continuing a solution in which four identical shielded Rankine vortices come into contact in a 4-symmetric annular configuration to enclose a core region of irrotational fluid. The pentapolar vortices observed in practice typically possess a core vortical region surrounded by four satellite vortical regions, usually with regions of irrotational fluid between the core and the satellites. We now attempt to construct a solution of the Euler equations which more closely models this situation. This example will serve as an illustration of the new methods.

Consider the situation in figure 2 in which four identical shielded Rankine vortices form an annular array (as considered in Crowdy (2001*a*)). These will model the satellite vorticity maxima of a pentapolar vortex. To model the core, another shielded Rankine vortex of smaller radius is placed inside the otherwise irrotational core so that it just touches all four of the satellite vortices. We suppose that the centre of this central shielded Rankine vortex corresponds with the coordinate origin.

The domain in figure 2 has a fourfold rotational symmetry about the origin. Because the vortices do not overlap, integration over the total domain is additive so that the mean value formula (4.1) can be used to show that such a domain satisfies the quadrature identity:

$$\iint_D h(z)\,\mathrm{d}x\,\mathrm{d}y = \pi h(z_1) + \pi h(z_2) + \pi h(z_3) + \pi h(z_4) + \pi(\sqrt{2}-1)^2 h(0), \tag{7.1}$$

where

$$z_1 = \sqrt{2}, \qquad z_2 = \mathrm{i}\sqrt{2}, \qquad z_3 = -\sqrt{2}, \qquad z_4 = -\mathrm{i}\sqrt{2}. \tag{7.2}$$

The algebraic curve corresponding to this quadrature domain happens to be easy to construct. It is given by

$$\mathcal{P}(z,\bar{z}) = 0, \tag{7.3}$$

where

$$\begin{aligned}\mathcal{P}(z,\bar{z}) = (|z|^2 - (\sqrt{2}-1)^2)(|z-\sqrt{2}|^2-1)(|z-\mathrm{i}\sqrt{2}|^2-1)&\\ \times (|z+\sqrt{2}|^2-1)(|z+\mathrm{i}\sqrt{2}|^2-1).&\end{aligned} \tag{7.4}$$

Now consider a generalized quadrature identity, parametrized by two real parameters (r, p), given by

$$\iint_D h(z)\,\mathrm{d}x\,\mathrm{d}y = \pi r^2 h(z_1) + \pi r^2 h(z_2) + \pi r^2 h(z_3) + \pi r^2 h(z_4) + \pi p^2 h(0), \tag{7.5}$$

where we have altered the coefficients of the quadrature identity (7.1) so as to be consistent with seeking connected quadrature domains sharing the same symmetries as the original configuration of touching discs. In § 7 *b* it will be seen that the parameter r is directly related to the strength of the satellite line vortices, while p is directly related to the strength of the central line vortex.

Using theorem 5.1 the associated matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} k & 0 & 0 & 0 & 4p^2 & 0 \\ 0 & g & 0 & 0 & 0 & -4 \\ 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \\ 4p^2 & 0 & 0 & 0 & -(4r^2+p^2) & 0 \\ 0 & -4 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.6}$$

where the last two rows and columns of (7.6) have been determined using a direct computation based on theorem 5.1. We have also taken into account the required symmetries of the quadrature domain; in particular, the fourfold rotational symmetries imply that all off-diagonal elements of the matrix $\boldsymbol{A}$ are zero except for those shown in (7.6). To see this, let $\omega = \mathrm{e}^{2\pi\mathrm{i}/4}$. Then, by the fourfold rotational symmetry of the domain, it is necessary that

$$\mathcal{P}(\omega z, \overline{\omega z}) = \mathcal{P}(z,\bar{z}), \tag{7.7}$$

so that the coefficients a_{kj} in (5.2) can be non-zero only if $k-j$ is an integer multiple of 4. The only possible cases are

$$k = j; \qquad k = 1, \quad j = 5; \qquad k = 5, \quad j = 1. \tag{7.8}$$

This leaves four (as yet undetermined) coefficients (e, f, g, k) needed to specify the algebraic curve. The values of these coefficients do *not* follow in any obvious way from the specification of the quadrature identity (7.5). The initial domain of touching discs corresponds to the parameter set

$$\left.\begin{array}{lll} r = 1; & p = \sqrt{2} - 1; & e = 10 - 8\sqrt{2}; \\ f = 2 - 4\sqrt{2}; & g = 13 - 8\sqrt{2}; & k = -3 + 2\sqrt{2}. \end{array}\right\} \tag{7.9}$$

These values are deduced from (7.4) by straightforward algebraic manipulations. We now seek to find domains for which the quadrature identity (7.5) holds with values of r and p close to, but different from, those given in (7.9) and such that the domain is now quintuply connected. Physically, this corresponds to a situation in which the five non-interacting shielded Rankine vortices of figure 2 have 'merged' to form a single compound (pentapolar) vortex.

Following the example of Gustafsson (1988), for a fourfold symmetric quintuply connected domain, we expect to find four special points on the rays

$$\arg[z] = \tfrac{1}{2}\pi l, \quad l = 0, 1, 2, 3; \tag{7.10}$$

and four others on the rays

$$\arg[z] = \tfrac{1}{4}(\pi + 2\pi l), \quad l = 0, 1, 2, 3. \tag{7.11}$$

The eight points of contact of the initial touching discs are to be found on these rays. Now define the functions

$$\left.\begin{array}{l} q_1(s) \equiv \mathcal{P}(s\mathrm{e}^{\mathrm{i}\pi/4}, s\mathrm{e}^{-\mathrm{i}\pi/4}), \\ q_2(t) \equiv \mathcal{P}(t, t), \end{array}\right\} \tag{7.12}$$

where parameters s and t are taken to be real. In order for s to represent the distance from the origin of the special point on $\arg[z] = \frac{1}{4}\pi$, and for t to represent the distance from the origin of the special point on the real axis, we require

$$\left.\begin{array}{r} q_1(s) = 0, \\ q_1'(s) = 0, \\ q_2(t) = 0, \\ q_2'(t) = 0. \end{array}\right\} \tag{7.13}$$

These equations are a result of conditions (6.1) and (6.2), which must hold at any special point of the quadrature domain. Equation (7.13) constitutes four real equations.

Recall that, in general, it should be expected that a quadrature domain with four enclosed irrotational regions should have four degrees of freedom corresponding to setting the area of each of the enclosed regions. Here, however, symmetry has been assumed so that all four irrotational regions must have the same area. We therefore only expect *one* degree of freedom associated with specifying the area of each of these regions. Let us assume that setting the real parameter s corresponds to fixing this degree of freedom.

Given the four equations (7.13) which relate (e, f, g, k) and (s, t) and assuming that r, p and s have been externally specified, five unknown parameters (e, f, g, k) and t remain to be determined. It is clear that we do not yet have enough equations to determine the required algebraic curve.

To fix an additional equation, we turn to the quadrature identity (7.5) and pick $h(z) = 1$. This yields the nonlinear equation

$$\mathcal{F}(e, f, g, k, t) = 0, \tag{7.14}$$

where the function $\mathcal{F}(e, f, g, k, t)$ is defined as

$$\mathcal{F}(e, f, g, k, t) \equiv \frac{1}{2\mathrm{i}} \oint_{\partial D} \bar{z}\, \mathrm{d}z - 4r^2 - p^2. \tag{7.15}$$

Note that the dependence of the function $\mathcal{F}$ on the parameter set (e, f, g, k, t) is rather subtle: it occurs only because the line integral in (7.15) is taken around the algebraic curve ∂D which depends on (e, f, g, k), which, through (7.13), themselves depend on t. Not all algebraic curves ∂D will be the boundaries of quadrature domains; indeed, only the special values of (e, f, g, k) which are the simultaneous solution of the five equations (7.13) and (7.14) are expected to give an algebraic curve bounding a quadrature domain.

It thus remains to solve the five equations (7.13) and (7.14) for (e, f, g, k) and t. To do this, it is useful to observe that the four equations (7.14) are linear in (e, f, g, k) (for given r, p, s and t). These four linear equations are solved for e, f, g and k as functions of r, p, s and t. These are substituted in (7.15), making it a single real nonlinear equation in one real unknown t. This single equation is solved using the following numerical scheme based on Newton's method.

(*a*) *Numerical method*

(i) Assuming that r, p and s are specified, a guess is made for t. Equations (7.13) are then used to solve for (e, f, g, k) in terms of r, p, s and t. This fixes some algebraic curve $\mathcal{P}(z, \bar{z}) = 0$ (although not necessarily the boundary of a quadrature domain).

(ii) By the reflectional symmetries of the configuration about the x- and y-axes, it is enough to consider one quadrant of the plane. Without loss of generality, the second quadrant is chosen. A highly accurate polynomial solver is used to compute real solutions x of $\mathcal{P}(x, x) = 0$. This provides the intersection points of the algebraic curve with the x-axis. The point x_0 furthest away from the origin is picked out.

(iii) The x-axis between $[x_0, 0]$ is discretized into N equally spaced points. Different values of N are used to ensure accuracy and convergence (but satisfactory results are obtained using $N = 10^5$). At each point x, a highly accurate (complex) polynomial solver is used to numerically compute all solutions y to the (real) polynomial $\mathcal{P}(x + \mathrm{i}y, x - \mathrm{i}y) = 0$. The solution set of 10 roots is filtered for real solutions y. The points at which the number of admissible (i.e. real) solutions (for y) changes are recorded (these correspond to turning points in y) and a small x-interval around such points is subdivided into 200 subintervals. (Because the curve is algebraic there is always only a finite number of such turning points.) This is a simple form of mesh refinement and is performed because, having chosen x as the independent coordinate, at such y-turning points, the change in y for a small change in x can be unacceptably large (unless we refine the mesh). This refinement is found to be crucial for accurate computations of the integral (see next step).

(iv) Once points on the curve are accurately determined, a trapezium rule is used to compute the one-dimensional complex integral in $\mathcal{F}(e, f, g, k, t)$. The mesh refinement in step (iii) is necessary to ensure that $|\mathrm{d}z|$ is always acceptably small in this numerical integration.

(v) Iterate on t until $\mathcal{F}(e, f, g, k, t) = 0$ in a Newton iteration.

For x_0 of order one, the global error in such a calculation is expected to be $\mathcal{O}(N^{-1})$, i.e. the error resulting from the numerical integration. Calculations were repeated using different values of N to ensure accuracy.

A useful check on the solutions obtained using this numerical method was made by choosing arbitrary analytic functions $h(z)$ (e.g. $h(z) = \mathrm{e}^z$, $1/(5 - z)$ were used) and numerically computing the integral of such $h(z)$ over the final domain by using Green's theorem to convert the integral to the one-dimensional integral

$$\frac{1}{2\mathrm{i}} \oint_{\partial D(t)} h(z)\bar{z}\,\mathrm{d}z, \tag{7.16}$$

and using the same integration procedure as described in the numerical scheme above. These numerically computed values were then directly compared with the values given by evaluating the right-hand side of the quadrature identity. It was deduced that a quadrature domain had been successfully reached when these two values of the integral agreed to within the accuracy of the numerical integration.

In figures 4 and 5, the values of e calculated using this numerical scheme are plotted as functions of the parameter s for fixed values of $r = 1.01$, $p = \sqrt{2} - 1$ (figure 4) and $r = 1.02$, $p = \sqrt{2} - 1$ (figure 5). The reason for plotting just the values of the coefficient e is because it turns out that the additional physical requirement that the five line vortices be stationary under the effects of the local non-self-induced velocity field only depends on this particular parameter (see § 7 *b*).

Note that the range of s values for which corresponding e values are plotted in figures 4 and 5 corresponds to the range of s for which quadrature domains exist for the fixed values of r and p. These ranges of existence are found to be given approximately by $s \in [0.92, 1.10]$ for the value $r = 1.01$ and $s \in [0.9, 1.15]$ for the value $r = 1.02$. These intervals are said to be 'approximate' because they are found simply by inspection of the corresponding domains plotted for different s values. It is found that as s draws towards the lower value of this range of existence, the special points move towards the enclosed boundaries of the domain and incipient cusps on these enclosed boundaries are seen as s reaches the lowest value of the interval of existence. As s reaches the highest value of the interval of existence, the special points draw close to the outer boundary of the domain and incipient cusps are seen

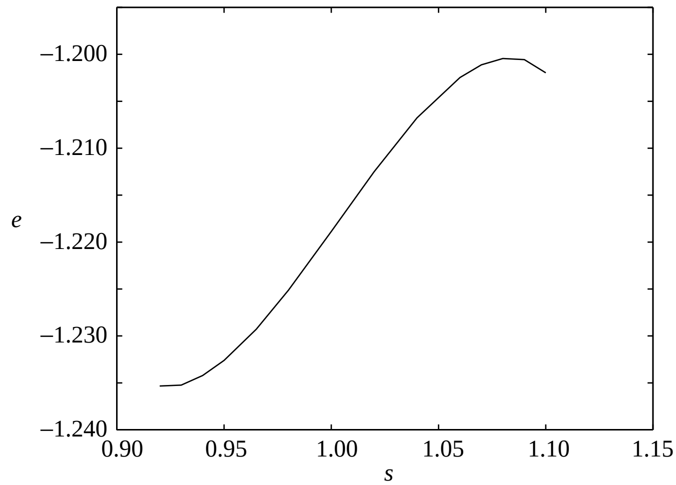

Figure 4. Graph of e against s for $r = 1.01$, $p = 0.4142$.

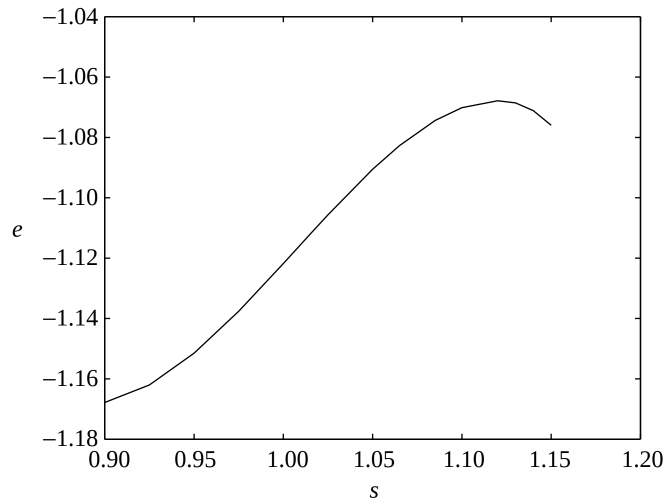

Figure 5. Graph of e against s for $r = 1.02$, $p = 0.4142$.

to develop on the outer boundary. These extreme cases turn out not to be relevant to the current application. As will be seen in the next section, only *one* choice of s in this range will give a global equilibrium of the Euler equation.

Other numerical methods than that described above can be used. Another method that can be used is to find a parametrization of the algebraic curve consisting of points equally spaced in arclength around the curve. A Nystrom-method based on the trapezium rule can then be used to compute integrals (this achieves superalgebraic convergence for smooth functions on periodic domains). For smooth curves with no points of large curvature, such a method requires much fewer points in the discretization of the curve. The numerical method described earlier is used here because the initial configurations display points of very high curvature (at the points of contact of the touching discs) and our method has no problem in dealing with such points. Furthermore, the method has the advantage of automatically locating the interior special points of the domains which are of particular interest in the present study.

(b) Stationarity of line vortices

For each of the values of e in either figure 4 or 5, there corresponds a quintuply connected quadrature domain. However, not all such quadrature domains are such that the five line vortices at points $z = 0$, $\pm\sqrt{2}$, $\pm i\sqrt{2}$ will be simultaneously stationary under the effects of the local non-self-induced velocity field. This is an imperative global equilibrium requirement for a consistent solution of the steady Euler equations.

To determine whether any of the quadrature domains just found do indeed correspond to exact multipolar equilibria of the Euler equations (and we emphasize that *a priori* there is no reason to expect that we should be able to find any) we observe that the function $S(z)$ appearing in (3.1) satisfies the equation

$$S(z) = \bar{z}, \tag{7.17}$$

on the boundary ∂D. However, it is also known that, on ∂D, $\bar{z}$ is a solution of

$$\mathcal{P}(z, \bar{z}) = 0. \tag{7.18}$$

We therefore conclude, following (5.6), that $S(z)$ satisfies the nonlinear equation of the form

$$p_5(z)[S(z)]^5 + p_4(z)[S(z)]^4 + p_3(z)[S(z)]^3 + p_2(z)[S(z)]^2 + p_1(z)S(z) + p_0(z) = 0, \tag{7.19}$$

for a set of polynomials $\{p_j(z)\}$ defined earlier. It is important to realize that (7.19) is now a relation between functions of z and therefore holds everywhere by analytic continuation. A local analysis of (7.19) reveals that, near $z = \sqrt{2}$, $S(z)$ has a Laurent expansion of the form

$$S(z) = \frac{\Gamma_s}{(z - \sqrt{2})} + \gamma_{s0} + \gamma_{s1}(z - \sqrt{2}) + \cdots, \tag{7.20}$$

where $\Gamma_s, \gamma_{s0}, \gamma_{s1}, \ldots$ are constants to be determined. The local velocity field has the form

$$u - iv = 2i\psi_z = 2i\bar{z} - 2iS(z) = 2i\left(\sqrt{2} - \frac{\Gamma_s}{(z - \sqrt{2})} - \gamma_{s0} + \mathcal{O}(|z - \sqrt{2}|)\right). \tag{7.21}$$

The condition that the line vortex at $z = \sqrt{2}$ is stationary under the effects of the non-self-induced terms in the local velocity field is therefore equivalent to the condition

$$\sqrt{2} - \gamma_{s0} = 0. \tag{7.22}$$

To determine γ_{s0} we analyse equation (7.19) locally near $z = \sqrt{2}$. Substituting (7.20) into (7.19) and examining the coefficient of $(z - \sqrt{2})^{-4}$ we obtain the relation

$$\Gamma_s = -\frac{p_4(\sqrt{2})}{p_5'(\sqrt{2})}, \tag{7.23}$$

while the coefficient of $(z-\sqrt{2})^{-3}$ yields the following equation for γ_{s0}:

$$\gamma_{s0} = -\frac{p_3(\sqrt{2}) + \Gamma_s p_4'(\sqrt{2}) + (\frac{1}{2}\Gamma_s^2)p_5''(\sqrt{2})}{5\Gamma_s p_5'(\sqrt{2}) + 4p_4(\sqrt{2})}. \tag{7.24}$$

It can easily be deduced that

$$\left.\begin{aligned} p_5(z) &= z^5 - 4z, \\ p_4(z) &= -(p^2+4r^2)z^4 + 4p^2, \\ p_3(z) &= ez^3, \end{aligned}\right\} \tag{7.25}$$

so that substitution into (7.23) and (7.24) yields

$$\Gamma_s = r^2. \tag{7.26}$$

This shows that specifying r in the quadrature identity amounts to specifying the strength of the satellite line vortices. Equation (7.24) together with (7.22) now provides the following condition on e that must be satisfied if we are to obtain a steady solution of the Euler equation:

$$e = 4p^2r^2 + 6r^4 - 8r^2. \tag{7.27}$$

By symmetry of the geometrical configuration and the associated velocity field, the three other line vortices at $z=-\sqrt{2}$, $\pm\mathrm{i}\sqrt{2}$ will also be stationary provided condition (7.27) is satisfied. It remains to ensure that the line vortex at $z=0$ is stationary. In fact, no additional conditions on the parameters are required because the central line vortex is automatically stationary. This can be argued using symmetry of the velocity field about the origin. Alternatively, the function $S(z)$ has the following expansion in the vicinity of the origin:

$$S(z) = \frac{\Gamma_c}{z} + \gamma_{c0} + \gamma_{c1}z + \cdots, \tag{7.28}$$

and the condition for stationarity of the central line vortex is that

$$\gamma_{c0} = 0. \tag{7.29}$$

That this condition is satisfied can be verified by performing an analysis analogous to that just described. Such an analysis also leads to the relation

$$\Gamma_c = p^2, \tag{7.30}$$

which shows that specifying the parameter p in the quadrature identity amounts to specifying the strength of the central line vortex.

It turns out that exact pentapolar equilibria of the Euler equations *can* indeed be found. From (7.27), the equilibrium values of e for the following three example choices of the pair (r,p) are computed (to three decimal places) to be

$$\left.\begin{aligned} r &= 1.01, \quad p = 0.4142, \quad e = -1.217; \\ r &= 1.02, \quad p = 0.4142, \quad e = -1.115; \\ r &= 1.01, \quad p = 0.4261, \quad e = -1.176. \end{aligned}\right\} \tag{7.31}$$

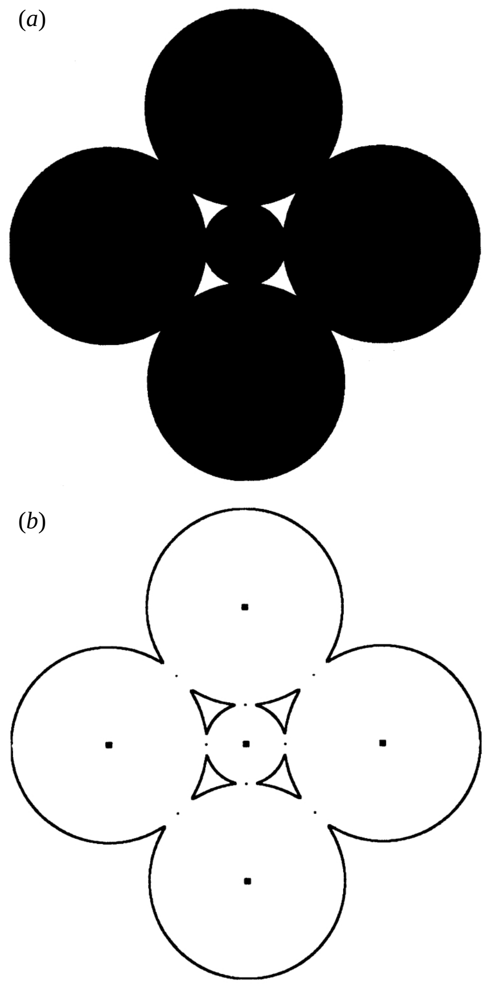

Figure 6. $r = 1.01$, $p = 0.4142$. (a) Region of non-zero vorticity. (b) Bold dots indicate point vortex positions and small dots indicate special points of the domain (or stagnation points of the flow).

It is found that there exist values of the parameter s (yielding quintuply connected quadrature domains) corresponding to these equilibrium values of e. These are found to be given, respectively, by

$$\left.\begin{aligned} s &= 1.006, \\ s &= 1.011, \\ s &= 1.008. \end{aligned}\right\} \tag{7.32}$$

The first two values can be found, in principle, by reading off the corresponding values of s from the graphs in figures 4 and 5. This fitting was in fact done numerically (by adapting the Newton methods already described) by *specifying* the value of e required for equilibrium (for a given (r, p)) and then finding the corresponding (s, f, g) required to give a quadrature domain.

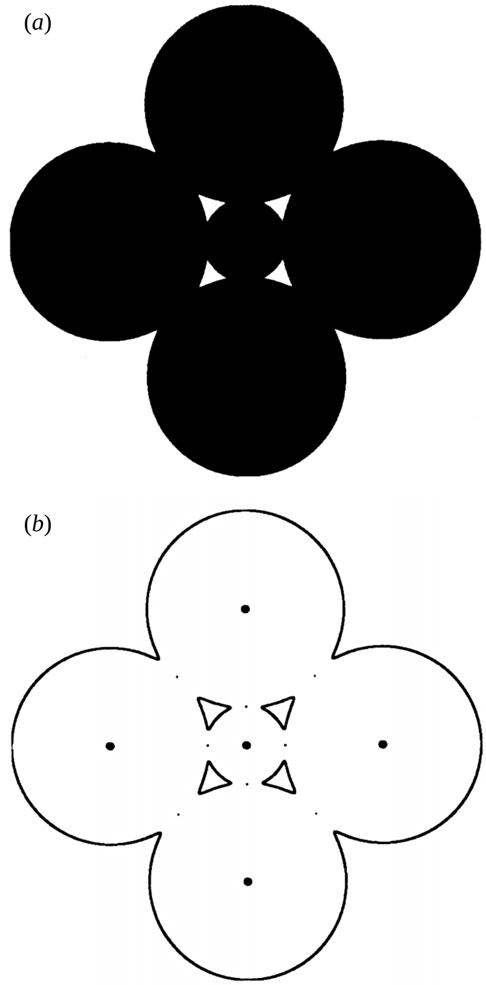

Figure 7. $r = 1.02$, $p = 0.4142$. (*a*) Region of non-zero vorticity. (*b*) Bold dots indicate point vortex positions and small dots indicate special points of the domain (or stagnation points of the flow).

In order to examine the shapes, the corresponding steady vortical structures are plotted in figures 6–8. Each figure corresponds to each of the three different choices of the parameters r and p and consists of two diagrams: the region of non-zero vorticity is shown as a shaded region in the first diagram, while the second diagram shows the same solution with the positions of the superposed line vortices and the special points clearly marked.

It might not be expected that these special points, which clearly have great *mathematical* importance (not least in the practical matter of constructing the domains, as has been seen), would have any *physical* significance. However, it turns out that they do. Consider the velocity field given by

$$u - \mathrm{i}v = 2\mathrm{i}\psi_z = 2\mathrm{i}(\bar{z} - S(z)). \tag{7.33}$$

(a)

(b)

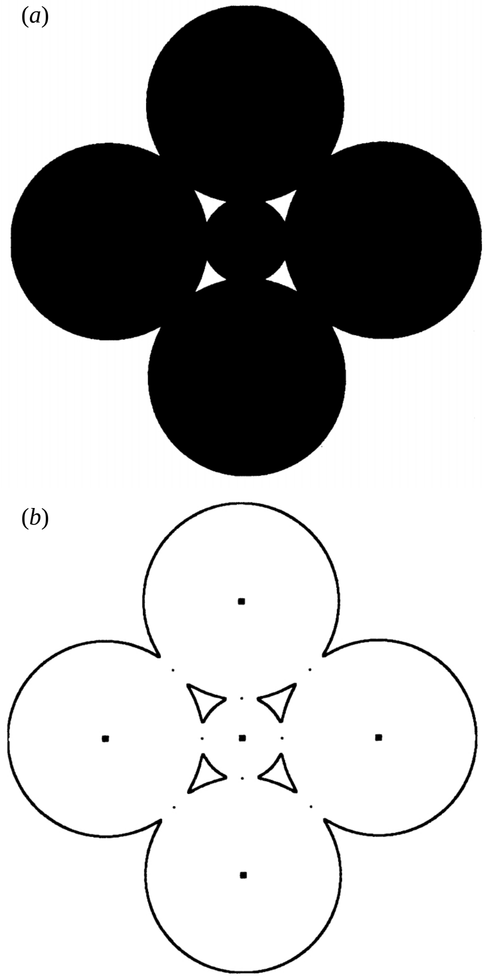

Figure 8. $r = 1.01$, $p = 0.4261$. (a) Region of non-zero vorticity. (b) Bold dots indicate point vortex positions and small dots indicate special points of the domain (or stagnation points of the flow).

Recall from (6.1) that, at all special points z_s in the set V_0,

$$\mathcal{P}(z_s, \bar{z}_s) = 0. \tag{7.34}$$

However, it is also true (from (7.19)) that the function $S(z)$ satisfies the equation

$$\mathcal{P}(z, S(z)) = 0, \tag{7.35}$$

which holds everywhere by analytic continuation. Evaluating (7.35) at the special point z_s inside the domain we obtain the relation

$$\mathcal{P}(z_s, S(z_s)) = 0. \tag{7.36}$$

But (7.34) and (7.36) are both valid at the point z_s so that together they imply that

$$\bar{z}_s = S(z_s), \tag{7.37}$$

which, using (7.33), implies that

$$u - \mathrm{i}v = 0 \quad \text{at } z_{\mathrm{s}}. \tag{7.38}$$

Thus the *special points* of the quadrature domain are precisely the *stagnation points* of the physical flow field inside the patch. These stagnation points have great physical importance because they are the points of confluence of the separatrix streamlines separating the core and satellite vortical regions. Note also that the parameter s now takes on a physical significance: it is the distance away from the central line vortex (which is at the origin) of four symmetrically disposed stagnation points of the flow.

It is interesting that in this particular physical application of the theory of quadrature domains there exists a physical interpretation of the mathematical notion of the 'special points' of a quadrature domain.

8. Summary and discussion

This paper has demonstrated how to construct geometrically complex global equilibria of the two-dimensional Euler equation by means of a 'nonlinear superposition' method in which a set of shielded Rankine vortices merge to form a compound multipolar equilibrium. The principle is very general. The construction has been implemented explicitly using the theory of quadrature domains and algebraic curves.

The study of multiple vortex equilibria of the Euler equations is important; point vortex models are the most analytically tractable models and Campbell & Ziff (1978) have catalogued stable and unstable equilibria for configurations consisting of up to 50 line vortices. It is relatively easy, from an analytical point of view, to superpose distributions of point vortices and to seek equilibria. In general, it is much more difficult (and in general impossible) to make any *analytical* progress in finding equilibria when regions of *distributed* vorticity, such as vortex patches, are superposed. This paper has presented some new ideas to show how this can be done when superposing shielded Rankine vortex solutions. With reference to the study of Campbell & Ziff (1978), Pullin (1992) has suggested that

> if all such equilibria can be continued to finite and perhaps different area and uniform vorticity for each vortex then the class of finite-area N-vortex equilibria is expected to be very dense, excluding even the possibility of further bifurcations off each branch for given N... the class of multiple-vortex equilibria is very large.

We believe that our results provide strong evidence to corroborate this statement, as well as providing a general constructive method for finding 'exact' representations of such equilibrium solutions.

A word of caution: it is important to point out that it is also possible to construct (in exactly the same way) a class of *quadruply connected* quadrature domains parametrized by (r, p) having three satellite vorticity maxima and three enclosed zones of irrotational fluid. It is reasonable to suppose that such quadrature domains might be used to construct solutions of the steady Euler equations modelling quadrupolar vortices. Indeed, this was attempted. However, while it is again found that the stationarity condition on the satellite line vortices imposes a condition on a parameter e depending on r and p (analogous to the condition (7.27)), it is found that for any chosen pair of parameters (r, p) considered, the value of e required for stationarity of the line vortices is always well outside the range of e for which quadrature domains (of this kind) exist. Therefore, no exact solutions of the steady Euler equations were found in this case. While this is not evidence of the non-existence of such solutions, it does highlight the non-trivial nature of the exact pentapolar solutions just found; even given that multiply connected quadrature domains can be explicitly constructed (a challenging task in itself), this is no guarantee that they will yield exact multipolar solutions of the steady Euler equations. Any such quadrature domains must satisfy additional constraints. A total balance of hydrodynamic pressure forces depending on the global geometry of the vorticity is required. The method presented here provides some analytical tools for studying when such global force balance can occur.

The emphasis of this paper has been a presentation of the mathematical construction of new solutions; however, the results give rise to many physical questions. For example, an important question concerns the linear and nonlinear stability of the solutions. Such questions are currently under investigation (Crowdy & Cloke 2001). It is an important feature of our solutions that they consist of combinations of line vortices and uniform vortex patches, which implies that their fully *nonlinear* evolution can be studied by straightforward adaptations of standard numerical contour dynamics algorithms (Pullin 1992).

Finally, we remark that the analytical ideas presented here can be extended in a slightly different direction to construct classes of steady *rotating* vortical equilibria (Crowdy 2001*b*). The *same* mathematical ideas therefore appear to lead to the construction of a wide range of *different* species of multipolar equilibria.

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