A New Approach to Free Surface Euler Flows with Capillarity

By Darren G. Crowdy

This article attempts to elucidate the underlying mathematical connection between the well-known exact solutions for the deep water capillary wave problem [G.D. CRAPPER, *J. Fluid Mech.*, 2:532–540 (1957)] and the recent discovery of a very special polar decomposition of solutions for a steadily translating bubble with surface tension [S. TANVEER, *Proc. Roy. Soc. A*, 452:1397–1410 (1996)]. This is achieved by describing a new and unified mathematical approach to the two separate physical problems. Using the new approach, Crapper's capillary wave solutions are retrieved in a novel and simplified fashion, while additional analytical insight into the nature of solutions for a steadily-translating bubble is obtained. The new approach is quite general and can also be used to obtain new exact results to other related free surface problems.

1. Introduction

In a recent paper, Tanveer [1] studied the problem of finding steady-state shapes for a two-dimensional (2-D) bubble, with surface tension on its boundary, translating at uniform speed in an infinite expanse of fluid. Mindful of the well-known exact solutions [2] for uniformly traveling pure capillary waves

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on deep water, Tanveer [1] attempted to develop a theoretical reformulation of the steadily translating bubble problem that was sufficiently general that it might reveal the existence of exact solutions for the steady-state shapes of the bubble. He did not quite succeed in this goal. However, using an inspired series of mathematical arguments, he deduced that the conformal mapping function taking the interior of the unit circle in a parametric ζ-plane to the simply connected fluid region exterior to the bubble must have the following very specific polar decomposition.

$$z_\zeta(\zeta) = \left(\frac{1}{\zeta} + \sum_{j=1}^{\infty} \frac{r_j \zeta}{\zeta^2 - \zeta_j^2} \right)^2 \tag{1}$$

This result, coupled with that of Crapper [2], prompt a natural question that is not addressed in Tanveer [1]: how exactly does the special polar decomposition of solutions (1) for a steadily translating bubble relate, mathematically, to the fact that exact solutions exist for pure capillary water waves on deep water? Moreover, are these intriguing analytical results indicative of a deeper mathematical structure underlying the general class of problems involving free surface potential flows with capillarity?

This article attempts to elucidate the mathematical connection between the results of Tanveer [1] and Crapper [2]. To do so, a unified approach to the general class of 2-D potential flow problems with surface tension on a free boundary is devised using a formulation in terms of conformal mappings and complex potentials. It is demonstrated herein that satisfying the free surface pressure condition is exactly equivalent to ensuring certain analyticity properties of the conformal mapping function z_ζ *inside* the unit circle ($|\zeta| \leq 1$), while simultaneously ensuring certain analyticity properties of a composite function $S(\zeta)$ (a nonlinear function of z_ζ and its conjugate function) *outside* the unit circle. The power of the new approach lies in the fact that it provides *necessary and sufficient* conditions on any candidate solution. Moreover, the form that any candidate solution must take can be deduced directly from the mathematical structure of the equations.

Using the new approach we retrieve the exact solution of Crapper [2] in a systematic and mathematically concise fashion. Once the theorems (providing the analyticity structure of the solution and the necessary and sufficient conditions to be satisfied by it) are established, the actual calculation of Crapper's solutions becomes trivial, and obviates the need for the extensive algebraic manipulation and special ansatz involved using the original method.

Crapper's [2] exact solution is shown herein to correspond to conformal maps, the derivatives of which are *rational functions* having a special analytic form. By contrast, we demonstrate explicitly why the related problem of a steadily translating bubble does *not* admit analogous rational function solutions. This provides complementary mathematical insight into the result of

Tanveer [1], who showed, using asymptotic arguments and comparison theorems, that, in fact, generic solutions have an infinity of poles. Furthermore, the new perspective enables us to deduce information on the nature of the solutions to the steadily translating bubble problem that is simply not available using purely local methods (e.g., asymptotics). Thus, we employ the theoretical reformulation to deduce global information on the zeros of the derivative of the conformal mapping function and, thereby, provide analytical insight into some numerical observations made by Tanveer [1].

One of the important aspects of the new approach is that it is very general. Indeed, it is appropriate to point out that the conceptual advantages afforded by the new approach presented here have already led to the unveiling of many new classes of exact solutions involving free surface flows with capillarity. These involve both simply and doubly connected fluid regions. The new solutions, derived using appropriate extensions of the general methods presented herein, are reported in full detail elsewhere [3, 4]. It is significant that these new exact solutions have not been observed previously despite their intimate mathematical connection (which becomes apparent from the present theoretical perspective) to the exact solutions found over 40 years ago by Crapper [2].

2. Mathematical formulation

In this article, we consider the following two classical fluid dynamical problems: (1) progressive, periodic capillary waves of fixed form on the surface of a fluid of infinite depth; and (2) steady uniform flow past a single bubble with surface tension on its boundary (equivalently, a steadily translating bubble).

Geometrically, the flow domains in each case are different but share the feature of being simply connected (at least, in a generalized sense described; for example, in [5, p. 125]. In both cases, infinitely far from the free surface, the flow is uniform. To treat both problems from a unified perspective, it is natural to exploit Riemann's Theorem and consider a conformal map to each flow domain from a standard domain in a parametric ζ-plane. In each case, we take the standard domain to be the interior of the unit-circle in the ζ-plane and denote the relevant conformal map as $z(\zeta)$.

For water waves, the form of the conformal mapping for a spatially periodic wave of wavelength $\lambda = (2\pi/k)$ (relative to a frame moving with the wave at a speed c to the right) is taken to be of the form

$$z(\zeta) = \frac{2\pi}{k} + \frac{i}{k}(\log \zeta + f(\zeta)) \tag{2}$$

where $f(\zeta)$ is taken to be analytic in the unit circle $|\zeta| = 1$ and is oblivious to the branch cut that has been explicitly factored out. The branch cut of

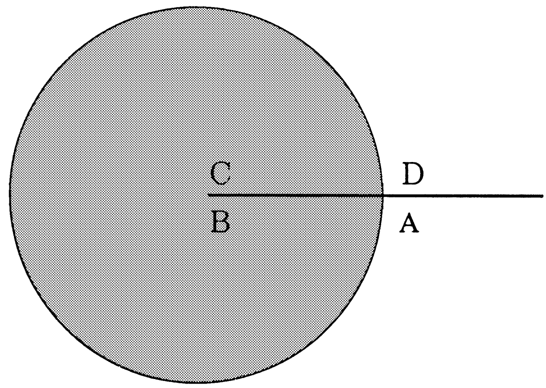

Figure 1. Parametric ζ-plane.

the logarithm is taken along the positive real axis, as shown in Figure 1. The corresponding physical plane is shown in Figure 2. The free surface of the traveling capillary wave corresponds to the unit circle $|\zeta| = 1$.

For the radial bubble geometry, the conformal mapping has the general functional form

$$z(\zeta) = \frac{a}{\zeta} + g(\zeta) \tag{3}$$

where the point $\zeta = 0$ maps to physical infinity, and $g(\zeta)$ is again analytic in $|\zeta| \leq 1$ for smooth bubble shapes, because it is required that the simple pole at $\zeta = 0$ be the only singularity of the conformal map inside the unit circle. In addition to these analyticity requirements, for a physically relevant solution, it is necessary that z be a univalent function in the unit circle. A necessary, but not sufficient, condition for this to be true is that z_ζ must have no zeros inside the unit circle. It is also assumed that the free boundaries in each case are smooth with no corners or cusps so that $z_\zeta \neq 0$ on $|\zeta| = 1$.

It is well-known that these free boundary problems can be reformulated as the problem of finding an appropriate conformal mapping function. In each problem, a complex potential $w(z)$ can be introduced. The complex potential is defined as

$$w(z) = \phi(x, y) + i\psi(x, y) \tag{4}$$

where $\phi(x, y)$ is a velocity potential and $\psi(x, y)$ is the stream function. The function $w(z)$ must be an analytic function of z everywhere in the fluid region. In the steady case, the kinematic boundary condition on the fluid interface is equivalent to specifying that

$$\text{Im}[w] = \psi = 0 \tag{5}$$

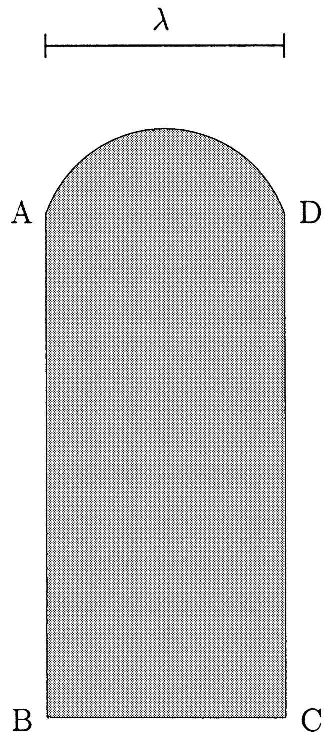

Figure 2. Physical plane—water waves.

on the boundary. In both problems, the boundary condition at physical infinity is that the flow velocity should be uniform. In the case of water waves, this corresponds to

$$w(z) \sim cz, \qquad \text{as} \qquad z \to \infty. \tag{6}$$

The two requirements (5) and (6) dictate the functional form for the composed function $W(\zeta) = w(z(\zeta))$. Specifically, for water waves it can be deduced that

$$W(\zeta) = \frac{2\pi c}{k} + \frac{ic}{k}\log\zeta \tag{7}$$

whereas, for the radial bubble case, we similarly deduce that $W(\zeta)$ necessarily has the form

$$W(\zeta) = B\left[\zeta + \frac{1}{\zeta}\right] \tag{8}$$

where B is a nondimensional parameter proportional to the speed of the fluid at infinity (with respect to a frame of reference in which the bubble itself is stationary). The functional forms (7) and (8) for $W(\zeta)$ are the *only* possible functional forms for analytic $W(\zeta)$ satisfying both (5) and the uniform flow conditions at infinity.

Remark 1: It will be clear that the general methods presented here are applicable, with suitable modifications, to any steady capillary flow in which the flow hodograph (i.e., $W(\zeta)$) is known, or can be deduced. For example, another problem amenable to a reformulation of this kind is that of finding steady solutions for a single bubble distorted by a steady infinite straining flow. This problem has been tackled numerically by Vanden–Broeck and Keller [6] and corresponds to

$$W(\zeta) = C\left(\zeta^2 + \frac{1}{\zeta^2}\right) \tag{9}$$

where C is a nondimensional parameter dependent on the strain-rate of the flow at infinity.

The nondimensionalized Bernoulli condition on the fluid interface can be written

$$\frac{1}{2}\left|\frac{dw(z)}{dz}\right|^2 + \kappa = \Gamma \tag{10}$$

where κ is the curvature of the interface, and Γ represents the Bernoulli constant. Rewriting this in terms of the conformal mapping variable ζ gives

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = -\frac{z_\zeta^{1/2}(\zeta)}{\bar{z}_\zeta^{1/2}(\zeta^{-1})}\mathrm{Re}\left[1 + \frac{\zeta z_{\zeta\zeta}}{z_\zeta}\right] + \Gamma z_\zeta \tag{11}$$

where we have used the fact that $\bar{\zeta} = \zeta^{-1}$ on $|\zeta| = 1$. This can more conveniently be written

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = -\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right]^{1/2} + \Gamma z_\zeta. \tag{12}$$

It will be assumed in what follows that $W(\zeta)$ is specified, as well as the value of Γ. For water waves, it is well-known that $\Gamma = c^2/2$, so we essentially specify the wavespeed c as well as the wavenumber k of the solution. In the bubble problem, specifying Γ corresponds to specifying the difference between the pressure in the bubble and the pressure at infinity. Specifying $W(\zeta)$ means specifying B (which is equivalent to specifying the speed of the fluid at infinity).

We now define some important functions:

DEFINITION 2.1. *Define the function* $S(\zeta)$ *as follows*:

$$S(\zeta) \equiv -\frac{d}{d\zeta}\left[\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right]^{1/2} + \Gamma z_\zeta(\zeta) \tag{13}$$

DEFINITION 2.2. *Define the function* $\Psi(\zeta)$ *as follows*:

$$\Psi(\zeta) \equiv \left(\frac{\zeta z_\zeta(\zeta)}{\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})}\right)^{1/2} \tag{14}$$

In addition, note that from (7) and (8), the quantity

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} \tag{15}$$

can be seen to be *analytic* outside the unit circle (including at the point at infinity) in *both* geometries. Its Laurent expansion, convergent in $|\zeta| \geq 1$, will be denoted

$$\frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} = \sum_{j=0}^{\infty} c_j \zeta^{-j}. \tag{16}$$

Finally, note that in the case of water waves, $c_0 = 0$; whereas, for the bubble, $c_0 \neq 0$. A finite subset of the coefficients $\{c_j\}$ will be important later in the development.

3. Singularity structure

By considering the form of the Bernoulli pressure condition, the following theorem can be established:

THEOREM 3.1. *In both the water wave and bubble problems* (*as described above*), *everywhere in the region* $|\zeta| \geq 1$, *the function* $\Psi(\zeta)$ *satisfies an equation of the form*

$$-\frac{d}{d\zeta}\Psi(\zeta) + q_1(\zeta)\Psi^2(\zeta) = q_2(\zeta) \tag{17}$$

where $q_1(\zeta)$ *and* $q_2(\zeta)$ *are analytic everywhere in the finite* ζ*-plane outside the unit circle*. *Specifically*,

$$q_1(\zeta) \equiv \frac{\Gamma\bar{z}_\zeta(\zeta^{-1})}{\zeta^2} \tag{18}$$

and

$$q_2(\zeta) \equiv \frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})} \tag{19}$$

Proof: The proof of this theorem is immediate using analytic continuation of the Bernoulli condition (12) into $|\zeta| \geq 1$ and by substituting the definition (14) for $\Psi(\zeta)$.

Remark 2: Note that the *form* of the equation in (17) for Ψ is that of a well-known Riccati equation if one views the coefficients q_1, q_2 as known functions (of course they are not, because they both depend implicitly on Ψ). Nevertheless, the important fact is that, provided a solution exists, q_1 and q_2 are known *a priori* to be analytic outside the unit circle.

THEOREM 3.2. *In both the water wave and bubble problems, the movable singularities of* $\Psi(\zeta)$ *in the finite* ζ*-plane outside the unit circle are necessarily simple poles.*

Proof: The differential equation satisfied by Ψ in $|\zeta| > 1$ is

$$-\frac{d\Psi}{d\zeta} + q_1(\zeta)\Psi^2 = q_2(\zeta). \tag{20}$$

Define a new variable $M(\zeta)$ via the standard Riccati transformation

$$\Psi = -\frac{1}{q_1}\frac{M_\zeta}{M} \tag{21}$$

for some $M(\zeta)$, then the equation for M in $|\zeta| > 1$ has the form

$$M_{\zeta\zeta} + q_3 M_\zeta + q_4 M = 0 \tag{22}$$

where

$$q_3 = -\frac{q_{1\zeta}}{q_1} \tag{23}$$

and

$$q_4 = -q_1 q_2. \tag{24}$$

Note, crucially, that q_3 and q_4 are known *a priori* to be analytic in $|\zeta| > 1$. Thus, it can be concluded that M has the same analytic structure (in $|\zeta| > 1$) as solutions of a second-order linear equation with analytic coefficients (and no fixed singularities). Using standard results from the theory of linear second-order differential equations [7, pp. 148–154], we deduce that M is analytic everywhere in $|\zeta| > 1$ (i.e., there are no fixed singularities in $|\zeta| > 1$, and linear equations allow no movable singularities). From (21), Ψ is, therefore, also analytic in $|\zeta| > 1$ except possibly at the zeros of M (recall that q_1 is analytic and does not vanish anywhere in $|\zeta| > 1$). Generically, the zeros of an analytic function are *simple*, which implies from (21) that generically, the singularities of Ψ are simple poles. Furthermore, from the differential equation (22), suppose that at a point ζ_0 (say) outside the unit circle, M has a *second-order* zero (so that $M(\zeta_0) = M_\zeta(\zeta_0) = 0, M_{\zeta\zeta}(\zeta_0) \neq 0$) then it can be seen from (22) that this can only be true provided q_3 has a simple pole, equivalently, provided that q_1 has a simple zero. However, this never occurs (in $|\zeta| > 1$) by the conformality constraints on the mapping function. Exactly the same arguments similarly prevent M from admitting zeros of order greater than 2 in $|\zeta| > 1$. It follows that the zeros of M in $|\zeta| > 1$ are necessarily simple, and, therefore, that the singularities of Ψ in $|\zeta| > 1$ are simple poles.

COROLLARY 3.1. *The singularities of* z_ζ *in the finite* ζ*-plane outside the unit circle are necessarily second-order poles*.

Proof: It is clear from the functional form of $\Psi(\zeta)$ that it has exactly the same singularity structure in the finite plane outside the unit circle as $R(\zeta)$. This is because $[\zeta^{-1}\bar{z}_\zeta(\zeta^{-1})]^{-1/2}$ is known to be analytic everywhere in the finite ζ-plane outside the unit circle. By Theorem 3.2, these singularities are necessarily simple poles. Squaring reveals that the singularities of z_ζ are then second-order poles.

Much information about the singularity structure of $z_\zeta(\zeta)$ everywhere in the finite ζ-plane has now been established. In summary, in both the water wave and bubble problems, generically, z_ζ is known to be analytic everywhere in $|\zeta| \leq 1$ except for known singularities at the origin and have only second-order poles everywhere in the finite ζ-plane in $|\zeta| > 1$. Any candidate solution z_ζ must possess these global analyticity properties.

4. Capillary waves

The following theorem shows that satisfying the Bernoulli condition on the free surface of the water wave is equivalent to ensuring certain global analyticity properties of the function $S(\zeta)$ in the extended complex plane *outside* the unit circle.

THEOREM 4.1. *The Bernoulli condition on the free surface of deep water capillary waves is equivalent to* $S(\zeta)$ *being analytic everywhere in* $|\zeta| \geq 1$ *with*

$$S(\zeta) \sim \frac{c_1}{\zeta}, \qquad \text{as} \qquad \zeta \to \infty \tag{25}$$

where c_1 *is defined in* (16).

Proof: First, assume that the Bernoulli condition holds on $|\zeta| = 1$. This implies

$$S(\zeta) = \frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})}. \tag{26}$$

By analytic continuation, this also holds off the unit circle. It is clear that this implies immediately that $S(\zeta)$ is analytic in $|\zeta| \geq 1$, and that $S(\zeta) \sim \frac{c_1}{\zeta}$.

Conversely, assume that $S(\zeta)$ is analytic outside the unit circle, including at infinity, where $S(\zeta) \sim \frac{c_1}{\zeta}$. Given these conditions on $S(\zeta)$, it is clear that it can be written in the form

$$S(\zeta) = \frac{H(\zeta)}{\bar{z}_\zeta(\zeta^{-1})} \tag{27}$$

for some $H(\zeta)$ (to be determined), which is analytic in $|\zeta| \geq 1$, and tends to a constant as $\zeta \to \infty$. This is because $\bar{z}_\zeta^{-1}(\zeta^{-1})$ is analytic outside the unit circle and $\sim \zeta$ as $\zeta \to \infty$. Note that

$$\overline{S(\zeta)\bar{z}_\zeta(\zeta^{-1})} = S(\zeta)\bar{z}_\zeta(\zeta^{-1}) \tag{28}$$

on $|\zeta| = 1$. This can be seen after some manipulation using the definition of $S(\zeta)$. (27) and (28) imply that $H(\zeta)$ is real on the unit circle, i.e.

$$\overline{H}(\zeta^{-1}) = H(\zeta) \tag{29}$$

on $|\zeta| = 1$. (29) furnishes the analytic continuation of $H(\zeta)$ into $|\zeta| \leq 1$ and, in particular, reveals that it is analytic everywhere in $|\zeta| \leq 1$. Thus, $H(\zeta)$ has been shown to be analytic everywhere in the finite plane, bounded as $\zeta \to \infty$ and real on the unit circle. By Liouville's theorem, $H(\zeta)$ is necessarily a real constant function. The additional condition that $S(\zeta) \sim \frac{c_1}{\zeta}$ sets this constant and finally implies that

$$H(\zeta) = \frac{c^2}{k^2} = W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1}) \tag{30}$$

(27) is, then, equivalent to the Bernoulli condition, and the theorem is proved.

Remark 3: It can further be shown that the aforementioned analyticity requirements on $S(\zeta)$ in $|\zeta| \geq 1$ are equivalent to a countably infinite set of line integral relations of the following form:

$$I[m] = \begin{cases} c_1 & \text{if} \quad m = 0 \\ 0 & m \geq 1 \end{cases} \tag{31}$$

where we define

$$I[m] \equiv \frac{1}{2\pi i} \oint_{|\zeta|=1} \zeta^{-m} S(\zeta) d\zeta. \tag{32}$$

It follows from Theorem 4.1. that this countable set of line integral relations is also equivalent to the Bernoulli condition on the free surface. This is significant, because the integral relations (31) are entirely analogous to a set of (different) integral relations, discovered originally by Longuet-Higgins [8] (see also Byatt-Smith [9] relevant to the study of deep water *gravity* waves, and, in the latter case, recognizing the equivalence of a set of line integrals to the Bernoulli condition has played a crucial role in the development of the theory of deep water gravity waves. It is interesting, therefore, to point out that the global reformulation of the capillary wave problem as presented in this article, is itself essentially equivalent to the consideration of a set of line integral relations.

THEOREM 4.2. (MAIN THEOREM). *The problem of periodic capillary water waves of wavelength* $(2\pi/k)$ *traveling at fixed speed* c *on deep water is equivalent to finding a function* $z(\zeta)$ *satisfying the following conditions*.

(1) $z(\zeta)$ *is a univalent conformal mapping from* $|\zeta| \leq 1$ *to the fluid domain of the form*

$$z(\zeta) = \frac{2\pi}{k} + \frac{i}{k}\Big(\log\zeta + f(\zeta)\Big), \tag{33}$$

where $f(\zeta)$ *is analytic in the unit circle*.

(2) $(\zeta z_\zeta)^{1/2}$ *has only simple pole singularities in the finite* ζ*-plane outside the unit circle*.

(3) *the function* $S(\zeta)$ *is holomorphic everywhere in* $|\zeta| \geq 1$ *with* $S(\zeta) \sim c_1\zeta^{-1}$ (*with* c_1 *as defined earlier*).

Proof: The proof follows from a combination of the preceding theorems.

Remark 4: Note that using a purely *local* analysis of the analytically continued Bernoulli equation for the water wave problem, it can be deduced that a *necessary* condition for solutions to the problem is that $S(\zeta)$ be analytic in the finite ζ-plane $|\zeta| \geq 1$. However, a local analysis says nothing about whether this is a *sufficient* condition. Indeed, the foregoing theorem clearly shows that the analyticity of $S(\zeta)$ in $|\zeta| \geq 1$ is necessary but *not* sufficient. However, by Theorem 4.2, sufficiency is provided by just a *single* extra requirement (that $S(\zeta) \sim c_1\zeta^{-1}$). The important point is that Theorem 4.2 provides *necessary and sufficient* conditions for a solution.

4.1. Rational function solutions

It is a well-known fact (see for example Hille [7, p. 105]) that certain Riccati equations admit rational functions with a finite number of poles as solutions. Given this fact and the fact that Ψ can be viewed as satisfying a Riccati-type equation outside the unit circle, we now ask a natural question: is it possible to find *rational* function solutions z_ζ to the water wave and bubble problems?; that is, solutions for ζz_ζ with only a *finite* number of second-order poles outside the unit circle. If so, the solutions will represent exact solutions to the problem in the sense that solutions will be expressible in terms of a finite set of parameters.

THEOREM 4.3. (WATER WAVES). *The water wave problem admits exact steady solutions of the form*

$$z_\zeta(\zeta) = \frac{iA}{\zeta}\left(\frac{\prod_{j=1}^{N}(\zeta - \eta_j)}{\prod_{k=1}^{N}(\zeta - \zeta_k)}\right)^2 \tag{34}$$

where $|\eta_j|,\ |\zeta_k| > 1$, *A is a real constant, and N is any integer* $N \geq 1$. *Moreover, the choice* $N = 1$, *yielding*

$$z_\zeta(\zeta) = \frac{iA}{\zeta}\left(\frac{(\zeta - \eta_1)}{(\zeta - \zeta_1)}\right)^2 \tag{35}$$

where A, η_1, *and* ζ_1 *are appropriate constants, corresponds to the exact solution obtained by Crapper* [2].

Proof: First, suppose that the only singularity of $\left(\zeta z_\zeta\right)^{1/2}$ outside the unit circle is one (and *only* one) simple pole at some finite (but as yet unknown) point ζ_1. This implies that z_ζ must have the form

$$z_\zeta(\zeta) = \frac{g(\zeta)}{\zeta(\zeta - \zeta_1)^2} \tag{36}$$

for some $g(\zeta)$ to be discussed shortly. For z_ζ that is a *rational function*, then, necessarily, the point at infinity is at worst a *polar singularity*. Application of the well-known *test-power test* (Hille [7]) shows that $S(\zeta)$ cannot possibly behave like (c_1/ζ) as $\zeta \to \infty$ for z_ζ having a polar singularity at the point at infinity. Therefore, in order that z_ζ be *analytic* as $\zeta \to \infty$, $g(\zeta)$ must be a polynomial of order at most 3. Note that, because $z_\zeta(\zeta)$ does not vanish *inside* the unit circle, the zeros of $g(\zeta)$ must be *outside* the unit circle. Furthermore, in order that $\Psi(\zeta)$ has no movable branch points outside the unit circle, it can further be deduced that necessarily, $g(\zeta) = A_1(\zeta - \eta_1)^2$ for some constants A_1 and some η_1 ($|\eta_1| \geq 1$) or that $g(\zeta) = A_2$—a constant function. Therefore, the only two admissible forms for z_ζ are

$$z_\zeta(\zeta) = \frac{A_1}{\zeta}\frac{(\zeta-\eta_1)^2}{(\zeta-\zeta_1)^2} \qquad \text{or} \qquad z_\zeta(\zeta) = \frac{A_2}{\zeta(\zeta-\zeta_1)^2} \tag{37}$$

for some constants $A_1, A_2, \eta_1, \zeta_1$. Note that z_ζ has *at most* three adjustable parameters.

Finding a solution to the capillary water wave problem is equivalent (by Theorem 4.2) to finding a conformal mapping function z_ζ analytic in $|\zeta| \leq 1$, except for a simple pole at $\zeta = 0$ so that $S(\zeta)$ is analytic everywhere outside the unit ζ-circle with the single additional requirement that $S(\zeta) \sim (c_1/\zeta)$ as $\zeta \to \infty$. These are now known to be *necessary and sufficient* conditions for a solution. With z_ζ of the forms given in (37) it is clear that the corresponding $S(\zeta)$ must have a *removable* second-order pole at ζ_1. The conditions that the principal part of $S(\zeta)$ vanishes at ζ_1 will, in general, impose two conditions on the parameters appearing in z_ζ. There is now just one other condition on $S(\zeta)$; namely, that

$$S(\zeta) \sim \frac{c_1}{\zeta} \qquad \text{as} \quad \zeta \to \infty. \tag{38}$$

Thus, in order for z_ζ of the forms in (37) to be solutions to the water wave problem, by the preceding theorems, there are just three conditions that it must satisfy.

It is clear that, in general, solutions of the first of the possible forms in (37) are possible, because it contains precisely three unknown parameters. In this case, the counting is consistent and in fact, the three nonlinear algebraic relations can be solved for A_1, η_1, and ζ_1, as will now be shown. By Theorem 4.2, the result necessarily constitutes a solution to the water wave problem provided only that it corresponds to a $z(\zeta)$ that is univalent in the unit circle.

Calculation of the solution ($N = 1$): For $N = 1$, it can be deduced that the solution has the form

$$z_\zeta(\zeta) = \frac{A_1}{\zeta}\frac{(\zeta-\eta_1)^2}{(\zeta-\zeta_1)^2}. \tag{39}$$

Using the rotational degree of freedom of the mapping theorem, we take ζ_1 to be real. To find a solution consistent with the required form (2) of the conformal mapping function it is clear that A_1 must be purely imaginary; i.e., $A_1 = iA$ for some *real* A. With z_ζ given in (39), the corresponding $S(\zeta)$ is given by

$$S(\zeta) = -i\frac{d}{d\zeta}\left(\frac{(\zeta-\eta_1)(1-\zeta\bar{\zeta}_1)}{(\zeta-\zeta_1)(1-\zeta\bar{\eta}_1)}\right) + \frac{i\Gamma A}{\zeta}\frac{(\zeta-\eta_1)^2}{(\zeta-\zeta_1)^2}. \tag{40}$$

The condition that the principal part of $S(\zeta)$ vanishes at $\zeta = \zeta_1$ provides the following two simple relationships between the parameters:

$$\eta_1 = -\zeta_1 \tag{41}$$

$$2A\Gamma = \frac{|\zeta_1|^2 - 1}{|\zeta_1|^2 + 1}. \tag{42}$$

The condition that $S(\zeta) \sim \frac{c_1}{\zeta}$ gives the third (and final) equation, which becomes (after utilizing (41) and (42))

$$\Gamma A^2 = \frac{|W_\zeta|^2}{2} = \frac{c^2}{2k^2}. \tag{43}$$

Integration with respect to ζ yields the map

$$z(\zeta) = \frac{2\pi}{k} + iA\left(\log\zeta - \frac{4\zeta_1}{\zeta - \zeta_1}\right) \tag{44}$$

where we have chosen the arbitrary integration constant to be consistent with (2). From (43) and the fact that $\Gamma = c^2/2$, it is clear that $A = 1/k$. This solution is exactly that obtained by Crapper [2] using very different methods.

Calculation for general N: In general, under the assumption that one seeks solutions in which z_ζ has N second-order poles, one can develop similar arguments to show that, necessarily, a possible exact solution is given by

$$[\zeta z_\zeta(\zeta)]^{1/2} = iA\left(\frac{\prod_{j=1}^{N}(\zeta - \eta_j)}{\prod_{k=1}^{N}(\zeta - \zeta_k)}\right) \tag{45}$$

where $|\eta_j|$, $|\zeta_k| > 1$, A is a real constant, and N is any integer $N \geq 1$. In this case, the corresponding $S(\zeta)$ will have removable second-order poles at each of the points ζ_k $(k = 1 \ldots N)$. The condition that the principal parts at each of the N points vanishes will impose $2N$ conditions on the parameters appearing in (45). In addition, there is but a further single condition that $S(\zeta) \sim (c_1/\zeta)$ as $\zeta \to \infty$. Thus, in general z_ζ must satisfy $2N + 1$ conditions in order to represent a solution. We note immediately, however, that z_ζ as given in (45) has $2N + 1$ adjustable parameters; namely,

$$\{\eta_j \mid j = 1 \ldots N\}, \qquad \{\zeta_k \mid k = 1 \ldots N\}, \qquad A. \tag{46}$$

Thus, in general, the "counting" for this program is consistent, and an exact solution of the form (45) can, in principle, be found by solving a nonlinear algebraic system of order $2N + 1$.

In fact, it turns out that a solution with this functional form *can* be found. For general N, a solution parametrized by ζ_1 (taken to be real) is readily found to be

$$\eta_j = e^{\frac{i\pi(2j-1)}{N}}\zeta_1, \qquad j = 1, 2, \dots N \tag{47}$$

$$\zeta_j = e^{\frac{2i\pi(j-1)}{N}}\zeta_1, \qquad j = 2, 3, \dots N \tag{48}$$

where

$$\Gamma A = \zeta_1 \prod_{j=1}^{N} \frac{(1-\zeta_1\bar{\zeta}_j)}{(1-\zeta_1\bar{\eta}_j)} \frac{\prod_{k=2}^{N}(\zeta_1-\zeta_k)}{\prod_{j=1}^{N}(\zeta_1-\eta_j)} \tag{49}$$

and

$$\Gamma A^2 = \frac{|W_\zeta|^2}{2} = \frac{c^2}{2k^2}. \tag{50}$$

These solutions also correspond to Crapper's solution [2], but with N periods of the wave described by the mapping function.

Remark 5: Note that Crapper's exact solution [2] becomes a natural consequence of Theorem 4.2. In particular, note that the actual calculation of the exact solution is trivial and involves none of the extensive algebraic manipulation needed using Crapper's original separation of variables method.

Remark 6: It is important to remark that the solution of the system must satisfy additional constraints stemming from the required univalency of z inside $|\zeta| \leq 1$. As is well-known, Crapper [2] observed that for sufficiently large amplitudes, the nonlinear wave begins to cross itself, thereby becoming inadmissible physically. This phenomenon can now be understood as a loss of univalency of a conformal mapping function as a second-order pole (at ζ_1) and zero (at $-\zeta_1$) of the derivative of the analytically continued mapping function get too close to the unit circle.

Remark 7: The above theorem says nothing about *uniqueness* of solutions. To the best of the author's knowledge, the question of uniqueness of Crapper's solution [2] is still open. It is believed that present mathematical arguments on the form of solutions might be made fully rigorous and perhaps then used to reduce the question of uniqueness of Crapper's solution [2] to the (more tractable, it is hoped) question of existence of additional solutions to the reduced finite nonlinear system. Such a proof of uniqueness does not yet seem to be available using alternative (e.g., functional analytical) techniques.

5. Steadily translating bubble

We now address, from the same unified perspective, the problem of a steadily translating bubble. In modifying the analysis, it will be clear that, in general, the approach can be adapted to any free surface problem with capillarity, provided that the flow hodograph (i.e., $W(\zeta)$) is known.

The direct analogue of Theorem 4.1 is:

THEOREM 5.1. *The Bernoulli condition on the free surface for a steadily translating bubble is equivalent to* $S(\zeta)$ *being analytic everywhere in* $|\zeta| \geq 1$ *with*

$$S(\zeta) \sim c_0 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2}, \qquad \text{as} \qquad \zeta \to \infty. \tag{51}$$

Proof: First, assume that the Bernoulli condition holds on $|\zeta| = 1$. This implies

$$S(\zeta) = \frac{W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1})}{2\bar{z}_\zeta(\zeta^{-1})}. \tag{52}$$

By analytic continuation, this also holds off the unit circle. It is clear that this implies immediately that $S(\zeta)$ is analytic in $|\zeta| \geq 1$ and that it behaves as in (51).

Conversely, assume that $S(\zeta)$ is analytic outside the unit circle, including at infinity, where it has the behavior given in (51). Given these conditions on $S(\zeta)$, it is clear that it can be written in the form

$$S(\zeta) = \frac{\zeta^2 H(\zeta)}{\bar{z}_\zeta(\zeta^{-1})} \tag{53}$$

for some $H(\zeta)$, which is analytic in $|\zeta| \geq 1$ and tends to a constant as $\zeta \to \infty$. This is because $\bar{z}_\zeta^{-1}(\zeta^{-1})$ is analytic outside the unit circle and $\sim \zeta^2$ as $\zeta \to \infty$. Note, however, that

$$\overline{S(\zeta)\bar{z}_\zeta(\zeta^{-1})} = S(\zeta)\bar{z}_\zeta(\zeta^{-1}) \tag{54}$$

on $|\zeta| = 1$. This implies that $H(\zeta)$ is real on the unit circle, i.e.

$$\frac{1}{\zeta^2}\overline{H}(\zeta^{-1}) = \zeta^2 H(\zeta) \tag{55}$$

on $|\zeta| = 1$. This equation furnishes the analytic continuation of $H(\zeta)$ into $|\zeta| \leq 1$ and, in particular, reveals that it is analytic everywhere in $|\zeta| \leq 1$ except for a fourth-order pole at $\zeta = 0$. Thus, we deduce that $\zeta^2 H(\zeta)$ is analytic everywhere in the finite plane, except for a second-order pole at both $\zeta = 0$ and $\zeta \to \infty$. It also satisfies (55). It can, therefore, be deduced that $\zeta^2 H(\zeta)$ necessarily has the general form

$$\zeta^2 H(\zeta) = d_2\zeta^2 + d_1\zeta + d_0 + \frac{\bar{d}_1}{\zeta} + \frac{\bar{d}_2}{\zeta^2} \tag{56}$$

for some d_0, d_1, d_2. Finally, the conditions (51) on the first three Laurent coefficients of $S(\zeta)$ fix the three constants d_0, d_1, and d_2 and force the following equation

$$\zeta^2 H(\zeta) = W_\zeta(\zeta)\overline{W}_\zeta(\zeta^{-1}) \tag{57}$$

(53) is then equivalent to the Bernoulli condition and the proof is then complete.

THEOREM 5.2. (MAIN THEOREM). *The problem of finding steady solutions for a steadily translating bubble is equivalent to finding a function $z(\zeta)$ satisfying the following conditions*:

(1) *$z(\zeta)$ is a univalent conformal mapping from $|\zeta| \leq 1$ to the fluid domain of the form*

$$z(\zeta) = \frac{a}{\zeta} + f(\zeta) \tag{58}$$

where $f(\zeta)$ is analytic in the unit circle.

(2) *$(\zeta z_\zeta)^{1/2}$ has only simple pole singularities in the finite ζ-plane outside the unit circle*.

(3) *The function $S(\zeta)$ is holomorphic everywhere in $|\zeta| \geq 1$ with*

$$S(\zeta) \sim c_0 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} \qquad \text{as} \qquad \zeta \to \infty \tag{59}$$

with c_0, c_1, and c_2 as defined in (16).

Proof: The proof follows from a combination of the preceding theorems.

Remark 8: It is clear from the preceding theorem that there are just two differences between finding solutions to this problem and finding solutions for pure capillary water waves. First, the singularity of the mapping function $z(\zeta)$ at $\zeta = 0$ is altered because of differences in geometry. Second, *three* of the Laurent coefficients of $S(\zeta)$ (valid in $|\zeta| \geq 1$) must be specified in contrast to the water wave case where just *one* such coefficient had to be specified. It will now be shown that these apparently minor differences have important consequences for the analytical structure of solutions in this case.

Remark 9: We note that it can again be shown that the Bernoulli condition on the surface of the bubble is further equivalent to a countable infinity of line integral relations, this time of the following form:

$$I[m] = \begin{cases} c_2 & \text{if} \quad m = 0 \\ c_1 & \text{if} \quad m = 1 \\ c_0 & \text{if} \quad m = 2 \\ 0 & m \geq 3 \end{cases} \tag{60}$$

where

$$I[m] = \frac{1}{2\pi i} \oint_{|\zeta|=1} \zeta^{-m+1} S(\zeta) d\zeta. \tag{61}$$

Note that this set of integral relations represents the generalization to the steadily translating bubble problem of the set of line integral relations originally found by Longuet–Higgins [8] and Byatt-Smith [9] relevant to the study of deep water *gravity* waves (see Remark 3). As mentioned earlier, in this latter context, this equivalent set of line integral relations has led to many theoretical advances. It is, therefore, reasonable to suggest that the generalized set (60) might similarly prove useful in unveiling important analytic information about the steadily translating bubble problem.

5.1. Nonexistence of rational function solutions

In previous sections on the water wave problem, a set of necessary and sufficient conditions on the analyticity structure of $z(\zeta)$ and $S(\zeta)$ was established. A particular class of rational function forms for z_ζ was seen to possess an analyticity structure that is consistent with the governing equations, and, in this case, the necessary and sufficient conditions on the analyticity structure of $S(\zeta)$ outside the unit circle reduces to a *finite* set of conditions on the zeros and poles of the rational function for z_ζ. This finite set of conditions was then found to be both consistent and solvable and to be exactly the set of necessary and sufficient conditions determining the solution.

In light of this, it is natural to expect that the *same* generic mathematical scenario occurs in the related problem of a steadily translating bubble problem; i.e., that the analogous conditions arising by hypothesizing a particular rational function form for the mapping function will again constitute a set of necessary and sufficient conditions that determine the solution. Under this assumption, it is natural to try to extend directly (to the translating bubble problem) the series of arguments leading to the exact rational function solutions to the water wave problem. However, as we now show, in this case the same methodology *fails* to unveil analogous rational function solutions to the steadily translating bubble problem. Nevertheless, new insight into the analytical structure of solutions can still be obtained.

Following the water wave analysis, suppose that we first seek a solution where the only singularity of z_ζ outside the unit circle is a single second-order pole at some finite point ζ_1. By arguments similar to those of a previous theorem, it is straightforward to deduce that, necessarily, z_ζ has one of three possible forms, the one with the greatest number of adjustable parameters being given by

$$z_\zeta(\zeta) = A\left(\frac{(\zeta - \eta_1)(\zeta - \eta_2)}{\zeta(\zeta - \zeta_1)}\right)^2 \tag{62}$$

where A and $|\eta_1|, |\eta_2|, |\zeta_1| > 1$ are constants. Clearly, there are at most four adjustable parameters in z_ζ. With z_ζ of this form, the corresponding $S(\zeta)$ will have a removable second-order pole at ζ_1. This leads to two conditions on the parameters of z_ζ. By Theorem 5.2, the only other requirements on $S(\zeta)$ for it to be a solution for a steadily translating bubble are the three conditions on the first three non-zero coefficients of the Laurent series of $S(\zeta)$. Therefore, in total, there will be five conditions on z_ζ. Unfortunately, there are at most four adjustable parameters in (62). The initial assumption that z_ζ has only one second-order pole outside the unit circle is, therefore, *inconsistent* in general.

It is clear that this argument is valid for a rational function of arbitrary degree n. Under the assumption that there are n second-order poles of z_ζ outside the unit circle, there will be $2n$ conditions arising from the requirement that the principal part of the corresponding $S(\zeta)$ vanishes at each pole ζ_j $(j = 1 \dots n)$ as well as three additional conditions on the first three non-zero Laurent coefficients of $S(\zeta)$. This leads to $2n + 3$ conditions. However, at each stage, z_ζ *necessarily* has a form with at most $2n+2$ adjustable parameters. Thus, we deduce that, generically, there cannot exist a finite distribution of second-order poles in the finite ζ-plane outside the unit circle for solutions to the steadily translating bubble problem. Hence, there are (generically) no rational function solutions to this problem.

Remark 10: We remark that this article is principally concerned with generic choices of Γ. It is appropriate to point out, however, that in the bubble problem, there are two special choices of Γ that are known to give exact solutions: one is the trivial circular solution; the other is the exact solution found originally by McLeod [10]. See Tanveer [1] for more details of these special cases.

5.2. An infinity of simple poles

The arguments above indicate that any hope of finding rational function solutions (with a *finite* number of poles) to the bubble problem is ill-founded. In fact, a full asymptotic analysis of the fixed singularity at $\zeta \to \infty$ of the

Riccati-type equation (16) shows that this point is, in fact, a cluster point of simple poles. That is, instead of a finite set of poles and zeros, solutions z_ζ to the problem of finding steady solutions for a translating bubble have a countably infinite number of poles and zeros and can be written as a pure Weierstrass product of the form

$$z_\zeta = \frac{A}{\zeta^2}\left(\frac{\prod_{j=-\infty}^{\infty}(\zeta - \eta_j)}{\prod_{k=-\infty}^{\infty}(\zeta - \zeta_k)}\right)^2 \tag{63}$$

where $|\zeta_k|, |\eta_j| \to \infty$ as $k, j \to \infty$. Tanveer [1] has essentially established this result using a related formulation of the problem. The result is perhaps not unexpected, because as $\zeta \to \infty$, (15) asymptotes to a *tangent equation* (Hille [7]) of the form

$$-\frac{d\Psi}{d\zeta} + p\Psi^2 = q \tag{64}$$

where p, q are non-zero constants. It is, therefore, not surprising that $\Psi(\zeta)$ (just like a tangent function) has an *infinity* of poles and zeros clustering at $\zeta \to \infty$ (the only fixed singular point of the Riccati-type equation outside the unit circle). The tangent function also possesses a pure Weierstrass decomposition. It is important for what follows to note that it can be shown that (63) is a *pure* limit point of poles and zeros with no additional singularity structure associated with the point at infinity. This fact is established by Tanveer [1].

5.3. *Information on the zeros of z_ζ*

Although we cannot find any exact rational function solutions, the new approach can, however, be used to make formal arguments that allow us to describe the analytical structure of solutions for this problem more sharply, in particular, with regard to its zeros. We now argue that solutions z_ζ to the problem of a steadily translating bubble, in fact, have the general infinite product representation

$$z_\zeta = \frac{A}{\zeta^2}(\zeta - a)^2(\zeta - b)^2\left(\frac{\prod_{j=-\infty}^{\infty}(\zeta - \eta_j)}{\prod_{j=-\infty}^{\infty}(\zeta - \zeta_j)}\right)^2 \tag{65}$$

where A, a, b, η_j, and ζ_j are constants.

Note the difference between (63) and (65): in (65), exactly *two* of the zeros of the mapping function (denoted a and b) have not been incorporated in the infinite product but have been written separately. This is important and has been done for reasons that will now be explained.

It is known that necessary and sufficient conditions for a solution derive from the conditions of vanishing principal parts of $S(\zeta)$ at any (and all) poles of z_ζ as well as three additional conditions on the first three nonvanishing coefficients of the Laurent expansion of $S(\zeta)$ valid in $|\zeta| > 1$. It has been further deduced that z_ζ has a pure Weierstrass decomposition of the form (63). For each pole, there will be two conditions arising from the vanishing principal part of $S(\zeta)$ at that point. From the point of view of "counting," these two conditions can be thought of as providing two equations to determine the pole position ζ_j and the position of a corresponding (or "associated") zero η_j.

There remain just three further conditions to be satisfied by the mapping function. One of these can be thought of as providing an equation for the normalization A. This leaves two further conditions. Although there are an infinite number of equations involved in this case, one still strives for consistency in the "counting" between the number of unknown parameters and the number of necessary and sufficient conditions satisfied by those parameters. Given that the mapping function has been deduced necessarily to have the form (63), the only possible way of introducing two extra degrees of freedom in z_ζ so that the two remaining conditions on $S(\zeta)$ can be satisfied is if z_ζ has *two additional zeros* not associated with any pole. These two additional zeros are denoted a and b in the representation (65).

The arguments above are formal; nevertheless, they are directly confirmed by the explicit numerical calculations of Tanveer [1]. Although Tanveer's primary objective was to compute the pole positions numerically, it is interesting that Tanveer [1] makes the following brief comment on the computed zeros of the mapping function $(z_\zeta)^{1/2}$.

> We do not list the zeros of $(z_\zeta)^{1/2}$ except to note that for $\gamma > 0$, all but two of them were found to be on the real ζ-axis in between the poles, which for $\gamma < 0$ shifted to the imaginary ζ-axis. In all cases, there was an additional pair of zeros along the imaginary ζ-axis (one on the positive and the other on the negative imaginary axis), which were the ones closest to the unit ζ-circle and appeared to have a dominating effect on the bubble shape close to pinching.

We surmise that the "additional pair of zeros" observed numerically in Tanveer [1] correspond to the "extra" zeros a and b in (65) (i.e., the two zeros not associated with any pole of the mapping function) the existence of which has been (formally) established above. The other zeros observed in Tanveer's numerical calculations correspond to those zeros "associated" with each pole ζ_j (these zeros are denoted η_j in the representation (65)). In this way, formal arguments based on the new approach are confirmed numerically. Thus, we have obtained complementary analytical insight into the structure of solutions that is not available using purely local methods. The positions of the poles and zeros of z_ζ nearest to the unit circle (for $\gamma < 0$ and $\gamma > 0$) are depicted in Figures 3 and 4 (according to the numerical computations per-

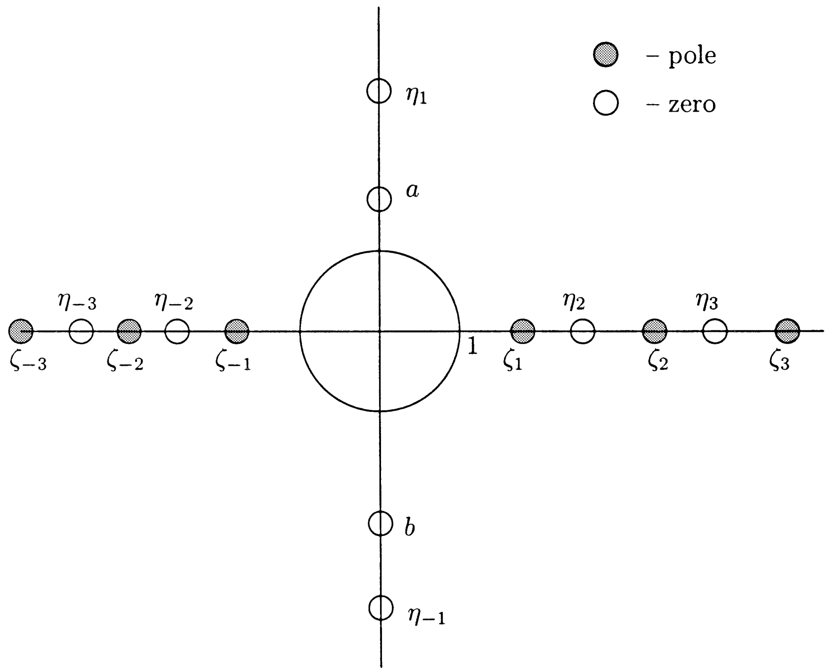

Figure 3. Zeros and poles for $\gamma < 0$.

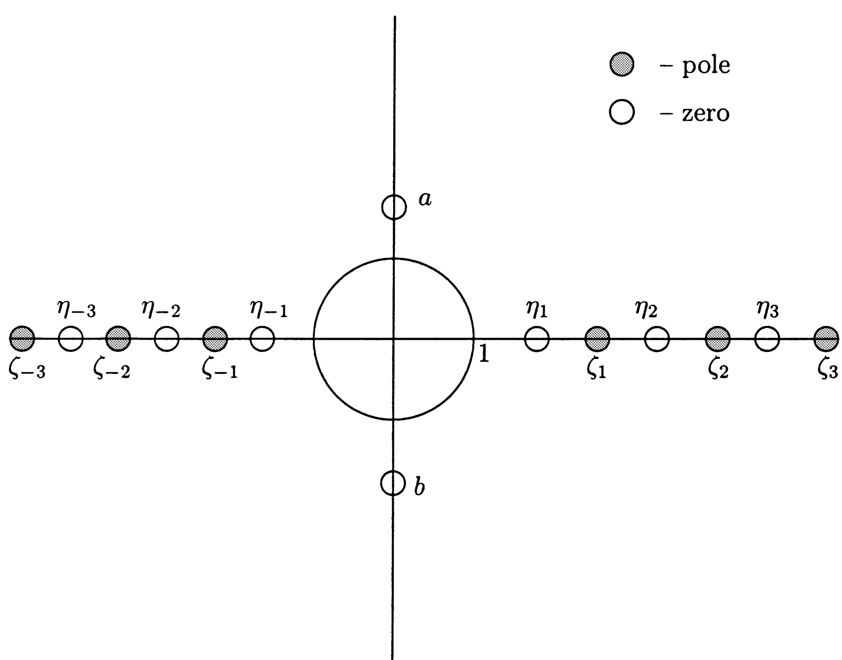

Figure 4. Zeros and poles for $\gamma > 0$.

formed by Tanveer [1]) along with their labels as they appear in the inferred representation given in (65).

This sharper description of the solution structure is important in that it provides a deeper understanding of the degrees of freedom in the (derivative of the) mapping function. This kind of information might be useful in devising effective numerical algorithms for computation of solutions based on the decomposition (65). It is conceivable that there exists some way to describe the infinity of pole and zero positions of (65) in closed form. If so, knowledge of this description would represent an exact solution to the problem. The necessary and sufficient conditions provided by the new approach and the implied global inter-relationships between the parameters in the mapping (65), might well lead to the identification of such a solution (if it exists).

6. Discussion

The goal of this article has been to present a new approach to the problem of free surface potential flows with capillarity with the express purpose of elucidating the mathematical connection between the analytical results of Crapper [2] and Tanveer [1]. By considering the singularity structure of an analytically continued boundary condition viewed as an ordinary differential equation of Riccati type, a unified framework in which to understand analytical properties of the two physical problems of pure capillary water waves and a steadily translating bubble with capillarity has been found. Using it, we gain insight into why it is natural to seek solutions that are rational functions and, in particular, why the water wave problem admits such solutions; whereas, the bubble geometry does not.

The new approach has indicated how the delicate interplay between the functional forms of $W(\zeta)$ and $z(\zeta)$ dictates whether or not (exact) rational function solutions exist. A natural question arises: do there exist other choices for $W(\zeta)$ and $z(\zeta)$, relevant to physical problems other than that of pure capillary water waves, that admit (exact) rational function solutions? We conclude by noting that the answer is in the affirmative and that the new approach developed here has unveiled several new classes of exact solutions to problems involving free surface potential flow with capillarity. These new solutions include both simply connected (Crowdy [3]) and doubly connected (Crowdy [4]) fluid regions. Full details of these new exact solutions, and the appropriate modifications of the present approach required to derive them, are reported elsewhere [3, 4].

Finally, we remark that the general methodology presented here can be generalized, in a natural way, to tackle the separate physical problem of multipole-driven Hele–Shaw flows with surface tension. Again, exact solutions to this problem can be found using very similar mathematical ideas to those developed here [11].

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