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A note on viscous sintering and quadrature identities

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Following the historical development of the theory of exact solutions for singularity-driven Hele-Shaw flows, this note demonstrates that the problem of two-dimensional viscous sintering preserves quadrature identities. This provides a unified theoretical perspective in which to understand the two separate free boundary problems. The result is established directly from the equations of motion without appeal to conformal mapping theory, although the result underlies the existence of exact conformal mapping solutions. The formulation leads to a concise, closed-form representation of the evolution equations for the parameters in the conformal mapping function. Some examples are given.

1 Introduction

In this paper, we demonstrate the relevance and usefulness of considering the linear functional $\mathscr{L}[h(z,t);D(t)]$, defined as

$$\mathscr{L}[h(z,t);D(t)] \equiv \int\int_{D(t)} h(z,t)dxdy, \tag{1.1}$$

(where $h(z,t)$ is an arbitrary function, analytic in the domain $D(t)$) in the problem of the viscous sintering of a fluid blob under the effects of surface tension.

The present note is motivated by the historical fact that it was essentially consideration of the linear functional (1.1) which led to the discovery of exact solutions for various singularity-driven flows in Hele-Shaw cells with zero surface tension as pioneered by Richardson [17, 18]. The latter problem has been compared with the sintering (Stokes) problem in the literature on a number of occasions [9, 5]. It would therefore seem to be of some theoretical interest to point out the relevance of the linear functional (1.1) to the viscous sintering problem, and this note is the first to do so explicitly, even though previous authors [4–6] have commented on the relevance (to the sintering problem) of certain 'moment-like quantities' (highly reminiscent of 'Richardson moments' [5] for the Hele-Shaw problem but defined in the ζ-plane, not the z-plane). This note demonstrates the result that 'Richardson moments' (defined in the z-plane and embodied in (1.1)) are, in fact, relevant to the theory of *both* the Hele-Shaw problem *and* the Stokes flow problem.

First, we define a *point differential functional of finite order* following the definition given in Davis [6]:

Definition The linear functional $\mathscr{L}$ defined on functions $h(z,t)$ that are analytic in D and continuous on $\partial D(t)$ will be said to be a *point differential functional of finite order* if it can be expressed in the following form:

$$\mathscr{L}[h(z,t);D(t)] = \sum_{n=1}^{N}\sum_{k=0}^{n_k} a_{nk}(t)h^{(k)}(z_n(t),t), \tag{1.2}$$

where $z_1(t), z_2(t), ..z_N(t)$ are a finite set of distinct points in the interior of D, $a_{nk}(t)$ are time-varying coefficients which are *independent of the function* $h(z,t)$ and $\{n_k \geqslant 0\}$ are integers. $h^{(k)}(z,t)$ denotes the k-th derivative with respect to z.

An expression of the form (1.2) constitutes a special class of *quadrature identities*. In a series of theorems, Davis [6] establishes the result that the conformal mapping $z(\zeta,t)$ from a unit-circle in a parametric ζ-plane and satisfying $z(0,t)=0$ to a region $D(t)$ is a rational function of ζ if and only if the linear functional $\mathscr{L}$ is a point differential functional of finite order. Thus, since it is has been formally proved using other methods [3] that the problem of viscous sintering is such that if the initial conformal map to a blob $D(0)$ is a rational function, then the conformal map $z(\zeta,t)$ to the blob $D(t)$ at later times will remain a rational function, it can be immediately deduced (from the theorems of Davis [6] just mentioned) that the problem of viscous sintering must be such as to preserve the class of quadrature identities (1.2).

In this note, we consider it of interest to *reverse* the above line of reasoning and prove *directly* from the equations of motion the fact that the viscous sintering problem preserves quadrature identities. This is done without appeal to conformal mapping theory. Then, by invoking the abovementioned theorems of Davis [6], it *follows* that the sintering problem preserves rational function conformal maps. In this way, we provide an independent proof that the problem of viscous sintering preserves rational function conformal maps while the line of reasoning leading to this result becomes *exactly analogous* to that which originally led to the identification of the exact solutions for the Hele-Shaw problem [17, 18]. This provides a certain unification in the method of solution for the two separate free boundary problems, and results in a certain theoretical uniformity of approach. Moreover, Richardson's approach to the Hele-Shaw problem has been generalized, over the years, in a number of ways (e.g. see Entov *et al.* [8, 7]). Understanding the sintering problem from the same theoretical perspective might lead to similar generalizations being made for the latter problem.

An important subsidiary result of this note is of practical value: it turns out that this approach leads to concise, closed-form representations of the relevant evolution equations for the exact solutions. As shown in a series of three examples, while it is difficult (or, at least, cumbersome) to explicitly write down the evolution equations for the poles and zeros of the rational function conformal map, an equivalent set of evolution equations under a 'nonlinear change of variables' resulting from the consideration of the linear functional $\mathscr{L}$ turn out to be rather concise.

2 A historical note

It is pertinent to point out the fact that the problem of free surface flows in a Hele-Shaw cell driven by a known distribution of sources and sinks has been found to admit exact rational function conformal mapping solutions essentially by consideration of the linear functional $\mathscr{L}$. We refer the reader to the early papers by Richardson [17, 18]. We note, however, that the existence of rational function exact solutions to this problem had been found much earlier [10–14] using various complex variable techniques, but not the 'Richardson moments' technique developed later (and independently) in by Richardson [17, 18]. The approach via 'Richardson moments' has proved to be a particularly useful one from a theoretical point of view and has led to many advances in the understanding of singularity-driven Hele-Shaw flows.

The (non-dimensionalized) mathematical problem considered by Richardson [17, 18] is to solve

$$\nabla^2 \phi = 0 \tag{2.1}$$

inside the fluid region, subject to the kinematic boundary condition

$$\mathrm{Im}[(z_t - (u + iv))\,\bar{z}_s] = 0, \tag{2.2}$$

where

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \tag{2.3}$$

and where s denotes an arclength parameter around the fluid boundary. The dynamic condition on the boundary ∂D is that

$$\phi = \text{constant} \quad \text{on} \quad \partial D, \tag{2.4}$$

while at each singularity z_j the complex potential $w(z) = \phi + i\psi$ must satisfy

$$w(z) \sim \frac{Q_j(t)}{2\pi} \log(z - z_j) \qquad \text{as} \quad z \to z_j. \tag{2.5}$$

It can be shown from the equations of motion (see Richardson [17] – although the notation is different) that the time derivative of $\mathscr{L}[h(z); D(t)]$ is given by

$$\frac{d\mathscr{L}[h(z); D(t)]}{dt} = \oint_{\partial D(t)} 2\phi_z h dz = \sum_j Q_j(t) h(z_j), \tag{2.6}$$

which can be directly integrated with respect to time to give

$$\mathscr{L}[h(z); D(t)] = \sum_j A_j(t) h(z_j), \tag{2.7}$$

where $\frac{dA_j(t)}{dt} = Q_j(t)$ and an integration constant has been taken to be zero (by choice of initial conditions). Thus, $\mathscr{L}[h(z); D(t)]$ is a point differential functional of finite order (at least for sufficiently short times $t > 0$).

Now, by invoking the theorems of Davis [6] mentioned earlier, it can be deduced that singularity-driven Hele-Shaw flows (with zero surface tension) are such as to preserve the rational function form of a conformal mapping function.

In the light of this, the aim of the present note is to establish these same results on the exact conformal mapping function solutions for the viscous sintering problem by directly shadowing this traditional approach to the Hele-Shaw problem. As shall be seen, it is a significantly more difficult mathematical problem (than in the Hele-Shaw problem just considered) to directly establish that the sintering problem preserves quadrature identities.

Therefore, the purpose of this note is to provide details of the proof. It will become clear that the sintering problem preserves quadrature identities for quite different mathematical reasons than in the singularity-driven Hele-Shaw problem.

3 Stokes flow due to surface tension (viscous sintering)

Consider the unsteady evolution of a general simply-connected plane blob of very viscous fluid evolving purely under the effects of surface tension. This problem has been considered, using conformal mapping techniques, by many previous authors [3–6,15–21] (and the references therein). In this section, we avoid the introduction of a conformal mapping function. Introducing a streamfunction $\psi(x, y)$ such that

$$\mathbf{u} = (\psi_y, -\psi_x), \tag{3.1}$$

then this streamfunction which satisfies a biharmonic equation in the fluid region, i.e.

$$\nabla^4 \psi = 0. \tag{3.2}$$

On the blob boundary, the stress condition is

$$-pn_j + 2e_{jk}n_k = \kappa n_j, \tag{3.3}$$

where e_{jk} are given by

$$e_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right). \tag{3.4}$$

Additionally, there is a kinematic boundary condition that the normal velocity V_n of a point on the boundary equals the normal fluid velocity at that point, i.e.

$$\mathbf{u.n} = V_n. \tag{3.5}$$

The general solution for the streamfunction can be written, at each instant of time t, as

$$\psi = \mathrm{Im}[\bar{z} f(z, t) + g(z, t)] \tag{3.6}$$

where $f(z, t)$ and $g(z, t)$ are analytic everywhere in the fluid region $D(t)$. The following relations can easily be established:

$$\begin{aligned} p - i\omega &= 4f'(z, t), \\ u + iv &= -f(z, t) + z\bar{f}'(\bar{z}, t) + \bar{g}'(\bar{z}, t), \\ e_{11} + ie_{12} &= z\bar{f}''(\bar{z}, t) + \bar{g}''(\bar{z}, t). \end{aligned} \tag{3.7}$$

It is known that the stress condition can be integrated once with respect to s to yield the following equation

$$f(z, t) + z\bar{f}'(\bar{z}, t) + \bar{g}'(\bar{z}, t) = -i\frac{z_s}{2}, \tag{3.8}$$

valid on the blob boundary where a constant of integration has, without loss of generality, been taken to be zero. The conjugate function $\bar{f}$ is defined as $\bar{f}(z) = \overline{f(\bar{z})}$. s is the arclength traversed in an anticlockwise direction around the blob. Finally, the kinematic condition (3.5) can be written as

$$\mathrm{Im}\,[(z_t - (u + iv))\,\bar{z}_s] = 0 \tag{3.9}$$

on the boundary of the blob, where z_t refers to the partial time derivative of z on the blob boundary keeping the arclength s fixed.

Theorem 3.1 *Under the evolution equations for Stokes flow, the following expression holds for the time evolution of* $\mathscr{L}$:

$$\frac{d}{dt}\mathscr{L}[h(z,t);D(t)] = \mathscr{L}[h_t(z,t) - 2f(z,t)h_z(z,t);D(t)], \tag{3.10}$$

where $h(z,t)$ *is analytic in* $D(t)$ *and continuous on* $\partial D(t)$ *and* $f(z,t)$ *is the Goursat function at time* t, *i.e.*

$$\psi = Im\,[\bar{z}f(z,t) + g(z,t)]\,. \tag{3.11}$$

Proof of Theorem 3.1 Using the complex form of Green's Theorem in the plane, it is clear that

$$\mathscr{L}[h(z,t);D(t)] = \frac{1}{2i}\oint_{D(t)} h(z,t)\bar{z}dz. \tag{3.12}$$

We now compute the time derivative of this quantity.

$$\frac{d\mathscr{L}[h(z,t);D(t)]}{dt} = \frac{1}{2i}\oint_{\partial D(t)} h_t\bar{z}dz + h_z z_t\bar{z}dz + h\,(\bar{z}dz)_t\,. \tag{3.13}$$

It is necessary to compute $(\bar{z}dz)_t$ from the equations of motion. The stress condition provides that

$$2\bar{f} + 2\bar{z}\frac{df}{dz} + 2\frac{dg}{dz} = i\bar{z}_s, \tag{3.14}$$

or equivalently

$$2\bar{f}dz + 2\bar{z}df + 2dg = i\bar{z}_s dz = ids, \tag{3.15}$$

using the fact that $z_s\bar{z}_s = 1$. Combining the stress condition with the kinematic condition yields

$$\mathrm{Im}\,[(z_t + 2f)\,\bar{z}_s] = -\tfrac{1}{2}, \tag{3.16}$$

or

$$z_t\bar{z}_s + 2f\bar{z}_s - \bar{z}_t z_s - 2\bar{f}z_s = -i, \tag{3.17}$$

which is equivalent to

$$z_t d\bar{z} + 2f d\bar{z} - \bar{z}_t dz - 2\bar{f}dz = -ids. \tag{3.18}$$

Combining (3.15) and (3.18) provides the expression

$$(\bar{z}dz)_t = 2dg + 2d(\bar{z}f) + \bar{z}dz_t + z_t d\bar{z} \tag{3.19}$$

which, when substituted into (3.13), gives the equation

$$\frac{d\mathscr{L}[h(z,t);D(t)]}{dt} = \frac{1}{2i}\oint_{\partial D(t)} h_t\bar{z}dz + 2hdg + 2hd(\bar{z}f) + [h_z z_t\bar{z}dz + hz_t\bar{z}dz + hz_t d\bar{z}]\,. \tag{3.20}$$

Note, however, that the terms in square brackets represent a total (spatial) differential, and therefore gives a zero total contribution. Note also that the second term on the right-hand side of (3.20) gives zero contribution, since both $h(z,t)$ and $g(z,t)$ are known to be analytic inside $D(t)$. Finally, using integration by parts, we obtain

$$\frac{d\mathscr{L}[h(z,t);D(t)]}{dt} = \frac{1}{2i}\oint_{\partial D(t)} (h_t - 2fh_z)\,\bar{z}dz \tag{3.21}$$

which, with a final application of Green's theorem, completes the proof. □

We now prove the fact (Theorem 3.2) that the equations of viscous sintering are such as to preserve quadrature identities. To do this, we define $\hat{z}(z,t)$ to be the solution of the complex partial differential equation

$$\frac{\partial \hat{z}}{\partial t} - 2f(z,t)\frac{\partial \hat{z}}{\partial z} = 0; \qquad \hat{z}(z,0) = z. \tag{3.22}$$

We assume that equation (3.22) can be solved for $\hat{z}(z,t)$ for times $0 < t < T$. Now let $h(z,t)$ be a given function of z and t. Given h, define a function $\hat{h}$ via the equation

$$\hat{h}(\hat{z}(z,t),t) = h(z,t), \tag{3.23}$$

which is assumed to hold at *all times* t. It will be assumed that the function $h(z,t)$ is completely arbitrary, except that it must be an analytic function of z everywhere in $D(0)$ and $D(t)$ for $0 < t < T$ for some non-zero T, a locally analytic (about $t = 0$) function of time, and also such that, for $0 < t < T$, the function $\hat{h}(z,t)$ as defined via (3.23) is an analytic function of z in $D(0)$. This last fact will be needed in the proof of Theorem 3.2.

We first prove two lemmas which will be useful later:

Lemma 3.1 *The following result is true:*

$$\left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)^n h(z,t) = \hat{h}^{(0,n)}(\hat{z}(z,t),t) \tag{3.24}$$

for $n \geqslant 1$, *where the notation* $g^{(m,n)}(-,-)$ *refers to the mth derivative with respect to the first argument and the nth derivative with respect to the second argument.*

Proof of Lemma 3.1 The proof is by induction. Consider the case $n = 1$:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)h(z,t) &= \left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)\hat{h}(\hat{z}(z,t),t)\\ &= \hat{h}^{(0,1)}(\hat{z},t) + \hat{h}^{(1,0)}(\hat{z},t)\left(\frac{\partial \hat{z}}{\partial t} - 2f(z,t)\frac{\partial \hat{z}}{\partial z}\right)\\ &= \hat{h}^{(0,1)}(\hat{z},t),\end{aligned} \tag{3.25}$$

where we have used (3.23) and (3.22). Now, suppose that the result is true for an arbitrary integer $n \geqslant 1$. Then

$$\left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)^n h(z,t) = \hat{h}^{(0,n)}(\hat{z},t), \tag{3.26}$$

which implies that

$$\left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)^{n+1} h(z,t) = \left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)\hat{h}^{(0,n)}(\hat{z}(z,t),t)$$
$$= \hat{h}^{(1,n)}\left(\frac{\partial}{\partial t} - 2f(z,t)\frac{\partial}{\partial z}\right)\hat{z}(z,t) + \hat{h}^{(0,n+1)}(\hat{z},t)$$
$$= \hat{h}^{(0,n+1)}(\hat{z},t), \tag{3.27}$$

by (3.22). The result follows for all integers $n \geqslant 1$ by induction. □

Lemma 3.2 *The following result is true for all integers* $k \geqslant 1$*:*

$$\hat{h}^{(k,0)}(\hat{z}(z,t),t) = \sum_{j=1}^{k} c_j(z,t)h^{(j,0)}(z,t), \tag{3.28}$$

where the coefficients $c_j(z,t)$ *are functions of the first* j *partial derivatives of* $\hat{z}(z,t)$ *with respect to* z*, i.e.*

$$c_j(z,t) = c_j\left(\frac{\partial \hat{z}}{\partial z}(z,t), \ldots, \frac{\partial^j \hat{z}}{\partial z^j}(z,t)\right). \tag{3.29}$$

Proof of Lemma 3.2 This straightforward result follows (again, by induction) by repeated (partial) differentiation of equation (3.23) with respect to z (at fixed t) and use of the chain rule. We omit the details. □

Theorem 3.2 *With the assumption that the boundary evolution is locally analytic in time, if the initial domain* $D(0)$ *is a domain (with an analytic boundary) such that for all* $h(z)$ *analytic in* $D(0)$

$$\int\int_{D(0)} h(z)\ dxdy = \sum_{n=1}^{N}\sum_{k=0}^{n_k} a_{nk}\frac{d^k}{dz^k}h(z)\bigg|_{\tilde{z}_n} \tag{3.30}$$

for some constants $\{a_{nk}\}$ *and* $\{\tilde{z}_n\}$ *(the points* $\{\tilde{z}_n\}$ *assumed strictly inside* $D(0)$*), then for sufficiently short times* $0 < t < T$*,* $D(t)$ *is also a domain for which*

$$\int\int_{D(t)} h(z,t)\ dxdy = \sum_{n=1}^{N}\sum_{k=0}^{n_k} d_{nk}(t)\frac{\partial^k}{\partial z^k}h(z,t)\bigg|_{z_n(t)}, \tag{3.31}$$

for some time-varying parameters $\{d_{nk}(t), z_n(t)\}$ *independent of* $h(z,t)$ *– any function analytic on* $D(t)$ *and continuous on* $\partial D(t)$*.*

Proof of Theorem 3.2 Assuming that the boundary evolution is locally analytic in time, $\mathscr{L}$ can be Taylor expanded about $t = 0$ for t within some radius of convergence T, i.e.

$$\int\int_{D(t)} h(z,t)\ dxdy = \int\int_{D(t)} h(z,t)\ dxdy\bigg|_{t=0} + t\frac{d}{dt}\int\int_{D(t)} h(z,t)\ dxdy\bigg|_{t=0}$$
$$+\frac{t^2}{2}\frac{d^2}{dt^2}\int\int_{D(t)} h(z,t)\ dxdy\bigg|_{t=0} + \ \ldots\ . \tag{3.32}$$

These derivatives can be computed by repeatedly using Theorem 3.1 and Lemma 3.1 above,

$$\int\int_{D(t)} h(z,t)\ dxdy = \int\int_{D(0)} \left(\hat{h}(z,0) + t\hat{h}^{(0,1)}(z,0) + \frac{t^2}{2}\hat{h}^{(0,2)}(z,0) + ...\right) dxdy$$
$$= \int\int_{D(0)} \hat{h}(z,t)\ dxdy, \quad (3.33)$$

after resumming the Taylor series expansion for $\hat{h}(z,t)$ about $t = 0$. Using (3.30) we obtain

$$\int\int_{D(t)} h(z,t)\ dxdy = \int\int_{D(0)} \hat{h}(z,t)\ dxdy = \sum_{n=1}^{N}\sum_{k=0}^{n_k} a_{nk}\hat{h}^{(k,0)}(\tilde{z}_n,t), \quad (3.34)$$

where we have used that fact that $\hat{h}(z,t)$ is assumed analytic in $D(0)$. Define the points $z_1(t), ..z_N(t)$ as the preimages, at time t, of the (fixed) points $\tilde{z}_1, ..\tilde{z}_N$ under the transformation defined by $\hat{z}(z,t)$, i.e.

$$\tilde{z}_n = \hat{z}(z_n(t),t); \quad n = 1, ..N. \quad (3.35)$$

It is clear that $z_n(0) = \tilde{z}_n$ and, for short enough times, it can always be assured that the points $z_1(t), ..z_N(t)$ are *inside* $D(t)$. Substituting into (3.34) we obtain

$$\int\int_{D(t)} h(z,t)\ dxdy = \sum_{n=1}^{N}\sum_{k=0}^{n_k} a_{nk}\hat{h}^{(k,0)}(\hat{z}(z_n(t),t),t). \quad (3.36)$$

Now substituting the result of Lemma 3.2 and using (3.23), we obtain

$$\int\int_{D(t)} h(z,t)\ dxdy = \sum_{n=1}^{N}\sum_{k=0}^{n_k} a_{nk}\hat{h}^{(k,0)}(\hat{z}(z_n(t),t),t)$$
$$= \sum_{n=1}^{N}\sum_{k=1}^{n_k} a_{nk}\sum_{j=1}^{k} c_j(z_n(t),t)h^{(j,0)}(z_n(t),t) + \sum_{n=1}^{N} a_{n0}h(z_n(t),t) \quad (3.37)$$

but, by rearrangement, this can be written in the form

$$\int\int_{D(t)} h(z,t)\ dxdy = \sum_{n=1}^{N}\sum_{k=0}^{n_k} d_{nk}(t)h^{(k,0)}(z_n(t),t), \quad (3.38)$$

where $d_{n0}(t) = a_{n0}$ and the remaining coefficients $d_{nk}(t)$ depend on the constants $\{a_{nk}\}$ and the time-evolving coefficient functions $\{c_j(z_n(t),t)\}$. □

Remark 3.1 *It is clear that* $d_{n0}(t) = a_{n0}$ *are invariants of the motion.*

4 Evolution equations

We now obtain an explicit set of first-order differential equations for the parameters $z_n(t),\ a_{nk}(t)$. This is most easily illustrated by means of a series of examples.

4.1 Example 1

Consider an initial blob for which $\mathscr{L}[h(z,0);D(0)]$ is given by the following point differential functional of finite order:

$$\mathscr{L}[h(z,0);D(0)] = \int\int_{D(0)} h(z,0)dxdy = \sum_{n=1}^{3} \tilde{a}_n h(\tilde{z}_n, 0), \tag{4.1}$$

where $h(z,0)$ is analytic in $D(0)$. The constants $\{\tilde{a}_j\}$ $\{z_j\}$ $(j = 1,2,3)$ are determined by the initial boundary shape. It is now known that the evolution of the blob is such that this linear functional remains a point differential functional of finite order, i.e.

$$\mathscr{L}[h(z,t);D(t)] = \int\int_{D(t)} h(z,t)dxdy = \sum_{n=1}^{3} a_n(t)h(z_n(t),t). \tag{4.2}$$

Differentiating (4.2) with respect to time gives

$$\frac{d\mathscr{L}[h(z,t);D(t)]}{dt} = \sum_{n=1}^{3} \dot{a}_n(t)h(z_n(t),t) + \sum_{n=1}^{3} a_n(t)\left[h^{(0,1)}(z_n(t),t) + \dot{z}_n(t)\ h^{(1,0)}(z_n(t),t)\right]. \tag{4.3}$$

However, from Theorem 3.1, it is also known that

$$\frac{d\mathscr{L}[h(z,t);D(t)]}{dt} = \mathscr{L}[h_t - 2f(z,t)h_z;D(t)] = \sum_{n=1}^{3} a_n(t)\ \left[h^{(0,1)}(z_n(t),t) - 2f(z_n,t)h^{(1,0)}(z_n,t)\right]. \tag{4.4}$$

However, (4.3) and (4.4) must be consistent, so that necessarily:

$$\dot{a}_j(t) = 0; \qquad j = 1,2,3,$$

$$\dot{z}_j(t) = -2f(z_j(t),t); \qquad j = 1,2,3. \tag{4.5}$$

The invariant quantities $a_j(t)$ are exactly those mentioned in the previous remark and can also be shown to be equivalent to those found (as line integrals quantities in the ζ-plane) in the ‘theorem of invariants’ of Crowdy & Tanveer [3].

4.2 Example 2

Consider now an initial blob $D(0)$ for which $\mathscr{L}[h(z,0);D(0)]$ is a point differential functional of the following form:

$$\mathscr{L}[h(z,0);D(0)] = \int\int_{D(0)} h(z,0)dxdy = \tilde{a}h(\tilde{z}_1,0) + \tilde{b}h^{(1,0)}(\tilde{z}_1,0), \tag{4.6}$$

where $\tilde{a},\tilde{b}$ and $\tilde{z}_1$ are determined by the initial shape. By Theorem 3.2, this linear functional remains a point differential functional under evolution, i.e.

$$\mathscr{L}[h(z,t);D(t)] = \int\int_{D(t)} h(z,t)dxdy = a(t)h(z_1(t),t) + b(t)h^{(1,0)}(z_1(t),t), \tag{4.7}$$

where, using the same methods as in the previous example, it can be shown that the equations for $z_1(t), a(t)$ and $b(t)$ are given by

$$\dot{z}_1(t) = -2f(z_1(t), t),$$

$$\dot{a}(t) = 0,$$

$$\dot{b}(t) = -2b(t)f_z(z_1(t), t), \tag{4.8}$$

with initial conditions $b(0) = \tilde{b};\ a(0) = \tilde{a};\ z_1(0) = \tilde{z}_1$.

4.3 Example 3

Finally, consider the initial quadrature identity given by

$$\mathscr{L}[h(z,0); D(0)] = \int\int_{D(0)} h(z,0)dxdy = \tilde{a}h(\tilde{z}_1, 0) + \tilde{b}h^{(1,0)}(\tilde{z}_1, 0) + \tilde{c}h^{(2,0)}(\tilde{z}_1, 0), \tag{4.9}$$

for some constants $\tilde{z}_1, \tilde{a}, \tilde{b}$ and $\tilde{c}$. Using the same methods as in the previous two examples, it can be shown that $D(t)$ evolves such that

$$\mathscr{L}[h(z,t); D(t)] = \int\int_{D(t)} h(z,t)dxdy = a(t)h(z_1(t), t) + b(t)h^{(1,0)}(z_1(t), t) + c(t)h^{(2,0)}(z_1(t), t), \tag{4.10}$$

where the relevant evolution equations are

$$\dot{z}_1(t) = -2f(z_1(t), t),$$

$$\dot{a}(t) = 0,$$

$$\dot{b}(t) = -2b(t)f_z(z_1(t), t) - 2c(t)f_{zz}(z_1(t), t),$$

$$\dot{c}(t) = -4c(t)f_z(z_1(t), t). \tag{4.11}$$

5 Conformal mapping parameters

It is worth noting that the evolution equations (4.5), (4.8) and (4.11) are very concise and conveniently stated. In the case of a simply-connected blob, it turns out that these evolution equations are sufficient to determine the evolution equations of the parameters appearing in the corresponding (rational function) conformal mapping function. To see this, consider Example 2. In this case, the conformal map has the form:

$$z(\zeta, t) = \frac{p(t)\zeta + q(t)\zeta^2}{(\zeta - \zeta_1(t))^2}. \tag{5.1}$$

This is the only rational function form for $z(\zeta, t)$ (with $z(0, t) = 0$) that provides the appropriate analyticity structure of the Schwarz function inside $D(t)$ (see Davis [6]). The evolution of the three parameters $p(t), q(t)$ and $\zeta_1(t)$ is required to determine the boundary evolution explicitly. To obtain their evolution, note the following 'nonlinear change of variables':

$$z_1(t) = z(\bar{\zeta}_1^{-1}(t), t),$$

$$a(t) = \mathscr{L}[1; D(t)],$$

$$b(t) = \mathscr{L}[z - z_1(t); D(t)]. \tag{5.2}$$

On the use of Green's theorem, and then the Residue theorem, in the linear functionals, it is clear that (5.2) provides an explicit (purely algebraic) system for determining $z_1(t), a(t)$ and $b(t)$ in terms of the conformal mapping parameters $\zeta_1(t), p(t)$ and $q(t)$.

Furthermore, note that it can be shown [3] that

$$2f(z) = -z_t + \zeta I(\zeta, t) z_\zeta, \tag{5.3}$$

where

$$I(\zeta, t) = \frac{1}{4\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left(\frac{\zeta' + \zeta}{\zeta' - \zeta} \right) \frac{1}{z_\zeta^{1/2}(\zeta', t) \bar{z}_\zeta^{1/2}(\zeta'^{-1}, t)}. \tag{5.4}$$

This provides an expression for $f(z)$ purely in terms of the conformal mapping. From this, it is easy to deduce that

$$2f_z(z) = \frac{-z_{\zeta t} + [\zeta I(\zeta, t) z_\zeta]_\zeta}{z_\zeta}. \tag{5.5}$$

In summary, equations (4.8) and (5.2) therefore provide a particularly concise, closed-form representation of the relevant evolution equations for the parameters in the conformal map (5.1). Indeed, (5.2) is a nonlinear change of variables in terms of which the evolution equations become particularly easy to write down explicitly.

6 Conclusion

The analysis here amplifies some recent observations of Cummings *et al.* [5], who comment on the appearance in the theory of both the Hele-Shaw and Stokes flow problem with zero surface tension of various 'moment-like' quantities, although it is pointed out by Cummings *et al.* [5] that the relevant moments in the Hele-Shaw problem are defined in the z-plane while those relevant to the Stokes flow problem are defined in the ζ-plane. This note demonstrates the important role played by the *same* linear functional $\mathscr{L}[h(z, t); D(t)]$ in the theory of both problems, and in particular, reveals that *both* free boundary problems can be tackled by considering 'Richardson moments' [5] defined in a z-plane.

In addition, understanding the viscous sintering problem using the notion of 'Richardson moments' also opens up the possibility that the various generalizations of Richardson's approach made by various authors (e.g. Entov *et al.* [8, 7]) with regard to the Hele-Shaw problem, might well now also be made to the Stokes flow problem.

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