M1M1: Progress Test 1 (2002): SOLUTIONS

1(a) Let

$$\begin{aligned} f(x) &= \left(\frac{1-\sin 3x}{1+\sin 3x}\right)^{1/2}, \\ &= \left(\frac{1-\sin 3x}{1+\sin 3x}\right)^{1/2}\left(\frac{1-\sin 3x}{1-\sin 3x}\right)^{1/2}, \\ &= \frac{1-\sin 3x}{(1-\sin^2 3x)^{1/2}}, \\ &= \frac{1-\sin 3x}{\cos 3x}, \\ &= \sec 3x - \tan 3x. \end{aligned}$$

Even part of $f(x)$ is $\sec 3x$, odd part is $-\tan 3x$.

(b) Note that

$$f(x+\frac{2\pi}{3}) = \left(\frac{1-\sin(3x+2\pi)}{1+\sin(3x+2\pi)}\right)^{1/2} = f(x),$$

so $f(x)$ is periodic with period $\frac{2\pi}{3}$.

2.(a) Note that $(1+x^3)^2 = 1+2x^3+x^6$, which is the required series expansion.

$$\begin{aligned} (b)\ (1+\exp(x))^3 &= 1+3\exp(x)+3\exp(2x)+\exp(3x) \\ &= 1+3\left(1+x+\frac{x^2}{2!}+...\right)+3\left(1+2x+\frac{4x^2}{2!}+...\right) \\ &\qquad +\left(1+3x+\frac{9x^2}{2!}+...\right) \\ &= 8+12x+12x^2+... \end{aligned}$$

(c) Note that

$$\begin{aligned} \frac{1}{1+e^x} &= \frac{1}{1+1+x+\frac{x^2}{2!}+} \\ &= (2+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...)^{-1} \\ &= \frac{1}{2}\left(1-\frac{x}{2}-\frac{x^2}{4}-\frac{x^3}{12}-...+\left(\frac{x}{2}+\frac{x^2}{4}+...\right)^2-\left(\frac{x}{2}+\frac{x^2}{4}+...\right)^3-... \right. \\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^3}{48}+... \end{aligned}$$

3. (a) On use of series expansions,

$$\lim_{x\to 0}\left(\frac{\sin x^{1/2}}{1-\cos x^{1/4}}\right) = \lim_{x\to 0}\left(\frac{x^{1/2}-\frac{x^{3/2}}{3!}+\ldots}{1-(1-\frac{x^{1/2}}{2!}+\ldots)}\right) = 2.$$

(b) By definition of $\sinh x$,

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sinh x}{1-e^x}\right) &= \lim_{x\to 0}\left(\frac{e^x-e^{-x}}{2(1-e^x)}\right)\\ &= \lim_{x\to 0}\left(\frac{1+x+\frac{x^2}{2!}+\ldots-(1-x+\frac{x^2}{2!}-\ldots)}{2(1-(1+x+\frac{x^2}{2!}+\ldots))}\right)\\ &= -1.\end{aligned}$$

(c) Note that

$$\lim_{x\to\infty}\left(\frac{1+3e^x+17e^{2x}}{1+5e^x+16e^{2x}}\right) = \lim_{x\to\infty}\left(\frac{17+3e^{-x}+e^{-2x}}{16+5e^{-x}+e^{-2x}}\right) = \frac{17}{16}.$$

(d) Note that

$$\begin{aligned}\lim_{x\to\infty}\left((e^x+1)^{1/2}-(e^x-1)^{1/2}\right) &= \lim_{x\to\infty}\left(e^{x/2}\left((1+e^{-x})^{1/2}-(1-e^{-x})^{1/2}\right)\right)\\ &= \lim_{x\to\infty}\left(e^{x/2}\left(1+\frac{e^{-x}}{2}-\frac{e^{-2x}}{8}+\ldots-(1-\frac{e^{-x}}{2}-\frac{e^{-2x}}{8}+\ldots)\right)\right)\\ &= \lim_{x\to\infty}\left(e^{-x/2}+\ldots\right) = 0.\end{aligned}$$

4. Let $y=\tanh^{-1}(x)$. Then

$$x = \tanh y = \frac{e^y-e^{-y}}{e^y+e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

On rearrangement,

$$e^{2y} = \frac{1+x}{1-x},$$

which implies that, provided $|x|<1$,

$$2y = \log\left(\frac{1+x}{1-x}\right) \quad \text{or} \quad \tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$