M1M1: Progress Test 1 (2003): SOLUTIONS

1.

$$
\begin{aligned}
f(x) &= \left(\frac{x+x^2}{x-x^2}\right)^{1/2} = \left(\frac{x+x^2}{x-x^2}\right)^{1/2}\left(\frac{x+x^2}{x+x^2}\right)^{1/2} \\
&= \left(\frac{(x+x^2)^2}{x^2-x^4}\right)^{1/2} \\
&= \left(\frac{|x|(1+x)}{|x|(1-x^2)^{1/2}}\right) \\
&= \frac{1}{(1-x^2)^{1/2}} + \frac{x}{(1-x^2)^{1/2}}.
\end{aligned}
$$

Therefore,

$$
f_e(x) = \frac{1}{(1-x^2)^{1/2}} \quad \text{and} \quad f_o(x) = \frac{x}{(1-x^2)^{1/2}}.
$$

It is clear that

$$
\begin{aligned}
f_e(\sin\theta) &= \frac{1}{(1-\sin^2\theta)^{1/2}} = \frac{1}{\cos\theta} = \sec\theta; \\
f_o(\sin\theta) &= \frac{\sin\theta}{(1-\sin^2\theta)^{1/2}} = \frac{\sin\theta}{\cos\theta} = \tan\theta.
\end{aligned}
$$

A well-known identity is that

$$
1 + \tan^2\theta = \sec^2\theta,
$$

so that

$$
1 + (f_o(\sin\theta))^2 = (f_e(\sin\theta))^2.
$$

2.(a) The partial fraction decomposition is

$$
\frac{x^4}{x^2+1} = \frac{x^4-1+1}{x^2+1} = \frac{(x^2-1)(x^2+1)+1}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}.
$$

(b) The factorization is

$$
x^4 + x^2 + 1 = (x^2+1)^2 - x^2 = (x^2+1+x)(x^2+1-x),
$$

so that the partial fraction decomposition has the form

$$
\frac{1}{x^4+x^2+1} = \frac{Ax+B}{x^2+1+x} + \frac{Cx+D}{x^2+1-x}.
$$

Some algebra give

$$\frac{1}{x^4+x^2+1} = \frac{1}{2}\left(\frac{x+1}{x^2+1+x} - \frac{x-1}{x^2+1-x}\right).$$

3. (a) Using the geometric series,

$$\begin{aligned}\frac{x^2+1}{x-1} &= -\frac{x^2+1}{1-x} = -(x^2+1)(1+x+x^2+x^3+...), \quad \text{for} \quad |x|<1 \\ &= -1-x-2x^2+...\end{aligned}$$

(b) On use of the exponential series

$$\begin{aligned}e^{\sin x} &= 1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+... \\ &= 1+\left(x-\frac{x^3}{3!}+...\right)+\frac{1}{2!}\left(x-\frac{x^3}{3!}+...\right)^2+... \\ &= 1+x+\frac{x^2}{2}+...\end{aligned}$$

(c) On use of the exponential series and geometric series,

$$\frac{1}{e^x+e^{-x}} = \frac{1}{2(1+X)} = \frac{1}{2}\left(1-X+X^2-X^3+...\right),$$

where

$$X = \frac{x^2}{2!}+\frac{x^4}{4!}+...$$

Substituting and expanding yields

$$\frac{1}{e^x+e^{-x}} = \frac{1}{2}-\frac{x^2}{4}+\frac{5x^4}{48}+...$$

4. (a) Let

$$y = \frac{2x+5}{x-1}$$

so that, on rearrangement,

$$x = \frac{y+5}{y-2}.$$

Then

$$f^{-1}(x) = \frac{x+5}{x-2}.$$

(b) Let

$$y = (e^x - e^{-x})^{1/2}$$

so that, on squaring this equation,

$$y^2 = e^x - e^{-x}.$$

Multiplication of this equation by e^x produces the following quadratic in e^x:

$$e^{2x} - y^2 e^x - 1 = 0,$$

which has the solution

$$e^x = \frac{y^2 + (y^4+4)^{1/2}}{2},$$

where we have discarded the other root because we need $e^x > 0$. Therefore,

$$x = \log\left(\frac{y^2 + (y^4+4)^{1/2}}{2}\right),$$

so that

$$f^{-1}(x) = \log\left(\frac{x^2 + (x^4+4)^{1/2}}{2}\right).$$