M1M1: Progress Test 1 (2004): SOLUTIONS

1. (a) Under the substitution $x \mapsto x + a$ with $a = 2\pi$, $f(x) \mapsto f(x)$ and there is no smaller value for which this is true. $f(x)$ is therefore periodic with period 2π.

(b) One could make use of the formulas

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2},$$

but it is easier to note that

$$\begin{aligned} f(x) &= \frac{\cos x}{1 - \sin x}\left(\frac{1 + \sin x}{1 + \sin x}\right), \\ &= \frac{\cos x + \cos x \sin x}{1 - \sin^2 x}, \\ &= \frac{\cos x + \cos x \sin x}{\cos^2 x}, \\ &= \sec x + \tan x. \end{aligned}$$

Therefore

$$f_o(x) = \tan x, \quad f_e(x) = \sec x.$$

(c) Clearly,

$$\frac{f_o(x)}{f_e(x)} = \frac{\tan x}{\sec x} = \sin x.$$

2. (a) First,

$$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1},$$

which is the final partial fraction form.

(b) Next,

$$\begin{aligned} \frac{x^2}{x^2-1} &= \frac{x^2-1+1}{x^2-1} = 1 + \frac{1}{x^2-1} \\ &= 1 + \frac{A}{x-1} + \frac{B}{x+1} \end{aligned}$$

for some A and B. Some algebra then leads to

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right),$$

which is the final partial fraction form.

(c) Finally,

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

for some A, B and C. Some algebra then gives

$$\frac{1}{x^3+1} = \frac{1}{3}\left(\frac{1}{x+1} + \frac{2-x}{x^2-x+1}\right).$$

3.(a) Range is $0 < f(x) < \infty$.

(b)

$$f(\log\theta) = e^{\log\theta} + 2e^{2\log\theta}.$$

But

$$e^{\log\theta} = \theta \quad \text{for} \quad \theta > 0$$

which means we need to solve

$$\theta + 2\theta^2 = 1 \quad \text{or} \quad 2\theta^2 + \theta - 1 = 0.$$

Solutions (by inspection or from the usual formula for the solutions of a quadratic) are

$$\theta = -1, \frac{1}{2},$$

but domain of definition of $\log\theta$ is $\theta > 0$ so only allowed solution is $\theta = \frac{1}{2}$.

(c)

$$y = e^x + 2e^{2x} \qquad \text{or} \qquad 2(e^x)^2 + e^x - y = 0.$$

On use of the formula for roots of a quadratic:

$$e^x = \frac{-1+\sqrt{1+8y}}{4},$$

where we have taken the +-root because we need $e^x > 0$. Therefore,

$$f^{-1}(x) = \log\left(\frac{-1+\sqrt{1+8x}}{4}\right).$$

4. (a) By definition,

$$\exp(x) = \cosh x + \sinh x,$$

so that

$$\exp(-x) = \cosh(-x) + \sinh(-x) = \cosh x - \sinh x.$$

Therefore,

$$1 = \exp(x)\exp(-x) = (\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x.$$

Alternatively, use the facts that

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}, \quad \sinh x = \frac{\exp(x) - \exp(-x)}{2}.$$

Therefore

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{\exp(x) + \exp(-x)}{2}\right)^2 - \left(\frac{\exp(x) - \exp(-x)}{2}\right)^2 \\ &= \frac{\exp(2x) + 2 + \exp(-2x)}{4} - \frac{\exp(2x) - 2 + \exp(-2x)}{4} \\ &= 1,\end{aligned}$$

which is the required identity.

(b) Dividing the identity in (a) by $\cosh^2 x$, yields the new identity

$$1 - \tanh^2 x = \text{sech}^2 x.$$