M1M1: Progress Test 1 (2005): SOLUTIONS

1. First note that

$$\exp(x+x^2) = \exp(x)\exp(x^2) = \exp(x^2)\,(\cosh x + \sinh x)$$

where we have decomposed $\exp(x)$ as a sum of its even part $(\cosh x)$ and its odd part $(\sinh x)$. Since $\exp(x^2)$ is even, we can read off that

$$f_e(x) = \exp(x^2)\cosh x, \quad f_o(x) = \exp(x^2)\sinh x.$$

Now,

$$\begin{aligned} f_e(x)^2 - f_o(x)^2 &= (\exp(x^2)\cosh x)^2 - (\exp(x^2)\sinh x)^2 \\ &= \exp(2x^2)\left(\cosh^2 x - \sinh^2 x\right) \\ &= \exp(2x^2) \end{aligned}$$

where we have used the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

2. The rational function is "top-heavy" so write

$$\begin{aligned} \frac{x^2}{x^2-1} = \frac{x^2-1+1}{x^2-1} &= 1 + \frac{1}{x^2-1} \\ &= 1 + \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}. \end{aligned}$$

(b) First note that

$$x^3 - 1 = (x-1)(x^2+x+1),$$

so the form of the partial fraction decomposition is

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

Some algebra leads to $A = 1/3, B = -1/3$ and $C = -2/3$.

3. Let

$$y = \frac{1}{\exp(x)+2}$$

so that, solving for x as a function of y yields

$$\exp(x) = \frac{1-2y}{y}.$$

Thus,

$$x = \log\left(\frac{1-2y}{y}\right),$$

or

$$f^{-1}(x) = \log\left(\frac{1-2x}{x}\right).$$

(b) The real function $\log x$ requires x to be real and strictly positive. In order that the quantity

$$\frac{1-2x}{x}$$

is real and strictly positive we need

$$0 < x < 1/2.$$

This is the required domain of the function.
(c) Noting that

$$f^{-1}\left(\frac{\sin\theta}{4}\right) = \log\left(\frac{1-\frac{\sin\theta}{2}}{\frac{\sin\theta}{4}}\right) = 0,$$

and the fact that the unique real solution of $\log x = 0$ is $x = 1$ means we require

$$\frac{1-\frac{\sin\theta}{2}}{\frac{\sin\theta}{4}} = 1.$$

Rearranging, this becomes

$$\sin\theta = \frac{4}{3}.$$

But $\sin\theta$ is well-known to be bounded between ± 1 for real θ, so there are no real solutions for θ.

4.(a) On use of the binomial series,

$$(1-x^2)^{1/3} = 1 + \frac{1}{3}(-x^2) + \frac{1}{2!}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(-x^2)^2 + ...$$

or

$$(1-x^2)^{1/3} = 1 - \frac{x^2}{3} - \frac{x^4}{9} + ...$$

(b) Noting that

$$\frac{\exp(x)}{2-x} = \frac{\exp(x)}{2}\left(1-\frac{x}{2}\right)^{-1}$$

and using the exponential series and the geometric series, we get

$$\frac{\exp(x)}{2-x} = \frac{1}{2}\left(1+x+\frac{x^2}{2!}+...\right)\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+...\right)$$

Expanding out product of the two series we get

$$\frac{\exp(x)}{2-x} = \frac{1}{2}+\frac{3x}{4}+\frac{5x^2}{8}+...$$

(c) By definition

$$\begin{aligned}
\text{sech } x = \frac{1}{\cosh x} &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+...\right)^{-1} \\
&= 1-\left(\frac{x^2}{2!}+\frac{x^4}{4!}+...\right)+\left(\frac{x^2}{2!}+\frac{x^4}{4!}+...\right)^2+... \\
&= 1-\frac{x^2}{2}+\frac{5x^4}{24}+...
\end{aligned}$$

where we have made use of the well-known geometric series expansion.