M1M1: Progress Test 2 (2002): SOLUTIONS

1.(a) $\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right) = \frac{d}{dx}\left(\frac{x^2-1+2}{x^2-1}\right) = \frac{d}{dx}\left(1+\frac{2}{x^2-1}\right) = -\frac{4x}{(x^2-1)^2}$

(or use product/quotient rule).

$$(b)\ \frac{d}{dx}\left(\frac{1}{x^x}\right) = \frac{d}{dx}e^{-x\log x} = e^{-x\log x}\left(-1-\log x\right) = -\frac{1+\log x}{x^x}.$$

(c) Let $y = \sinh^{-1}(x)$ so that $\sinh y = x$. Then

$$\cosh y \frac{dy}{dx} = 1,$$

which yields

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

(d) Note that

$$\frac{d(\sinh x)^{-1}}{dx} = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = -\frac{\cosh x}{\sinh^2 x}.$$

2. The rational function $f(x)$ can be written as

$$f(x) = 1 + \frac{x^2+1}{x-2} = x + 3 + \frac{5}{x-2}.$$

Seeking stationary points,

$$\frac{df}{dx} = 1 - \frac{5}{(x-2)^2}.$$

This equals zero when $x = 2 \pm \sqrt{5}$ so that stationary points are at $(2 \pm \sqrt{5}, 5 \pm 2\sqrt{5})$.

There is a vertical asymptote at $x = 2$.

As $x \to \pm\infty$, $f \to x + 3$, so these are also asymptotes. See Figure 1.

(b) Let

$$r = \frac{1}{\sqrt{\cos^4\theta - \sin^4\theta}} = \frac{1}{\sqrt{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}} = \frac{1}{\sqrt{(\cos^2\theta - \sin^2\theta)}}.$$

Therefore

$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1 \Longleftrightarrow x^2 - y^2 = 1.$$

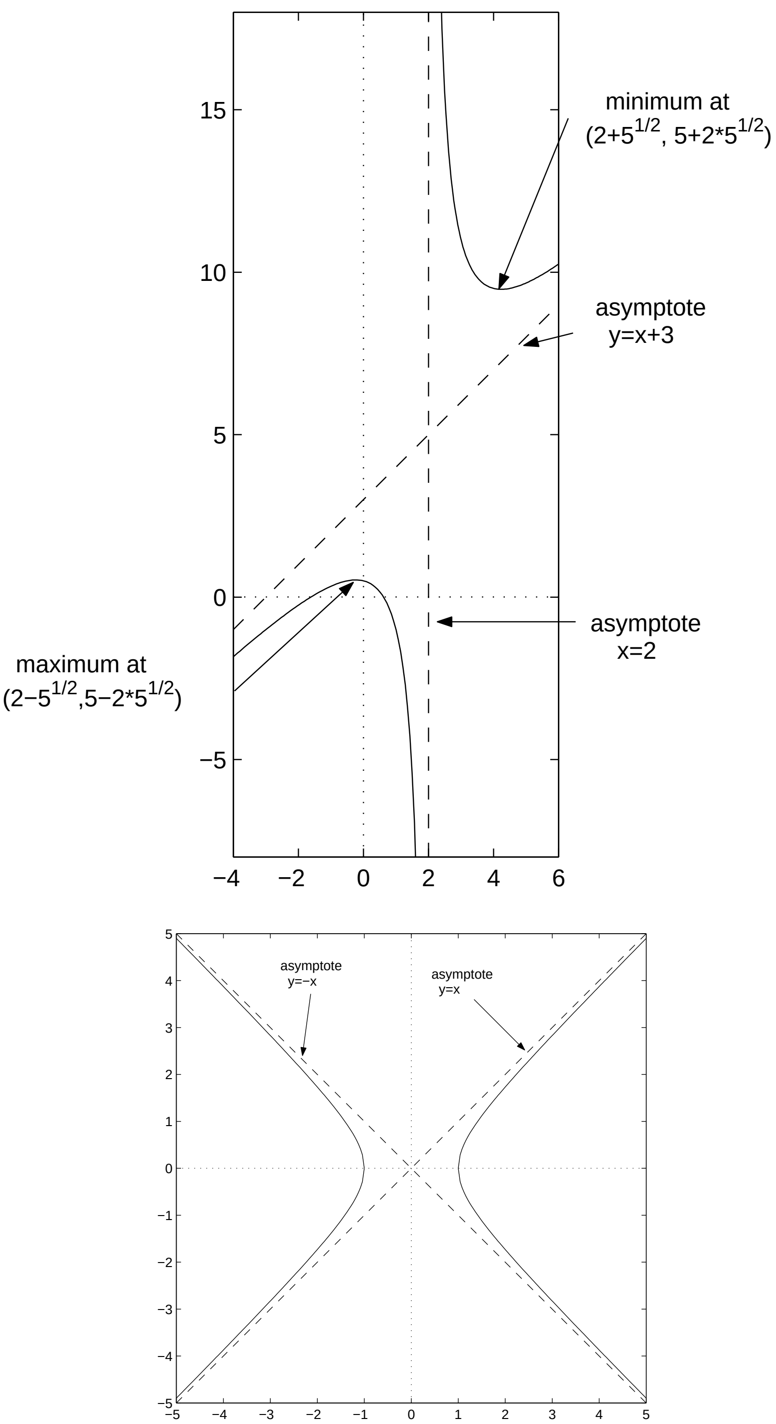

Figure 1: Graphs for 2(a) and 2(b)

The graph is therefore a hyperbola with asymptotes $y = \pm x$. See Figure 1.

3. Defining

$$f(x) = \log(1 + \sin x)$$

then, by differentiation,

$$\begin{aligned} f'(x) &= \frac{\cos x}{1 + \sin x}, \\ f''(x) &= -\frac{\sin x}{1 + \sin x} - \frac{\cos^2 x}{(1 + \sin x)^2}, \\ f'''(x) &= -\frac{\cos x}{1 + \sin x} + \frac{3 \cos x \sin x}{(1 + \sin x)^2} + \frac{2 \cos^3 x}{(1 + \sin x)^3}. \end{aligned}$$

Therefore, $f(0) = 0, f'(0) = 1, f''(0) = -1$ and $f'''(0) = 1$. Finally, on use of the Taylor series formula,

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

4. One method is to note that

$$\frac{d^n}{dx^n}\left(\frac{1}{(1-x)(2-x)}\right) = \frac{d^n}{dx^n}\left(\frac{1}{1-x} - \frac{1}{2-x}\right)$$

and use the facts that, for $n \geq 0$,

$$\frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = \frac{n!}{(1-x)^{n+1}}, \quad \frac{d^n}{dx^n}\left(\frac{1}{2-x}\right) = \frac{n!}{(2-x)^{n+1}}.$$

Therefore, evaluating at $x = 0$,

$$\left.\frac{d^n}{dx^n}\left(\frac{1}{(1-x)(2-x)}\right)\right|_{x=0} = n!\left(1 - \frac{1}{2^{n+1}}\right).$$

Alternatively, use the Leibniz rule:

$$\begin{aligned} \frac{d^n}{dx^n}\left(\frac{1}{(1-x)(2-x)}\right) &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \frac{d^j}{dx^j}\left(\frac{1}{1-x}\right) \frac{d^{n-j}}{dx^{n-j}}\left(\frac{1}{2-x}\right) \\ &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \frac{j!}{(1-x)^{j+1}} \frac{(n-j)!}{(2-x)^{n-j+1}} \end{aligned}$$

where the last line follows from the derivatives found above. Evaluating this at $x = 0$ gives

$$\frac{d^n}{dx^n}\left(\frac{1}{(1-x)(2-x)}\right)\bigg|_{x=0} = \frac{n!}{2^{n+1}}\sum_{j=0}^{n} 2^j.$$

But

$$\sum_{j=0}^{n} 2^j = \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1,$$

so that

$$\frac{d^n}{dx^n}\left(\frac{1}{(1-x)(2-x)}\right)\bigg|_{x=0} = \frac{n!}{2^{n+1}}\left(2^{n+1}-1\right) = n!\left(1-\frac{1}{2^{n+1}}\right).$$