M1M1: Progress Test 2 (2003): SOLUTIONS

1.(a) $\lim_{x\to 0}\left(\frac{\tanh x}{x}\right) = \lim_{x\to 0}\left(\frac{\sinh x}{x}\cosh x\right) = \lim_{x\to 0}\left(\frac{(x+\frac{x^3}{3!}+...)}{x}\cosh x\right) = 1.$

(b) Letting $x = \pi/2 + \epsilon$,

$$\lim_{x\to\pi/2}\left(\frac{\sin x - 1}{\cos x}\right) = \lim_{\epsilon\to 0}\left(\frac{\sin(\pi/2+\epsilon)-1}{\cos(\pi/2+\epsilon)}\right).$$

But

$$\sin(\pi/2+\epsilon) = \sin(\epsilon)\cos(\pi/2) + \sin(\pi/2)\cos\epsilon = \cos\epsilon,$$
$$\cos(\pi/2+\epsilon) = \cos(\epsilon)\cos(\pi/2) - \sin(\pi/2)\sin\epsilon = -\sin\epsilon,$$

so limit becomes

$$\lim_{\epsilon\to 0}\left(\frac{\cos\epsilon - 1}{-\sin\epsilon}\right) = \lim_{\epsilon\to 0}\left(\frac{(1-\frac{\epsilon^2}{2!}+...)-1}{-(\epsilon-\frac{\epsilon^3}{3!}+...}\right) = \lim_{\epsilon\to 0}\left(\frac{-\frac{\epsilon}{2!}+\frac{\epsilon^3}{4!}+...}{-(1-\frac{\epsilon^2}{3!}+...)}\right) = 0.$$

(c) Letting $\epsilon = \sqrt{x}$,

$$\lim_{x\to 0}\left(\frac{\sqrt{1+\sqrt{x}}-\sqrt{1-\sqrt{x}}}{\sqrt{x}}\right) = \lim_{\epsilon\to 0}\left(\frac{(1+\epsilon)^{1/2}-(1-\epsilon)^{1/2}}{\epsilon}\right)$$
$$= \lim_{\epsilon\to 0}\left(\frac{(1+\frac{\epsilon}{2}+...)-(1-\frac{\epsilon}{2}+...)}{\epsilon}\right) = 1.$$

2. Let

$$\log(1+x) = a_1x + a_2x^2 + a_3x^3 + ...$$

Then, from the identity,

$$\begin{aligned} 1+x &= \exp(a_1x + a_2x^2 + a_3x^3 + ...) \\ &= 1 + (a_1x + a_2x^2 + a_3x^3 + ...) + \frac{1}{2!}(a_1x + a_2x^2 + a_3x^3 + ...)^2 \\ &\quad + \frac{1}{3!}(a_1x + a_2x^2 + a_3x^3 + ...)^3 + ... \\ &= 1 + a_1x + \left(a_2 + \frac{a_1^2}{2}\right)x^2 + \left(a_3 + \frac{2a_1a_2}{2!} + \frac{a_1^3}{3!}\right)x^3 + ... \end{aligned}$$

Equating coefficients of same powers of x:

$$\begin{aligned} a_1 &= 1; \\ a_2 + \frac{a_1^2}{2} &= 0; \\ a_3 + \frac{2a_1a_2}{2!} + \frac{a_1^3}{3!} &= 0. \end{aligned}$$

Solving these yields $a_1 = 1, a_2 = -\frac{1}{2}$ and $a_3 = \frac{1}{3}$ so that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ...$$

$$\begin{aligned} (a)\ \lim_{x\to 0}\left(\frac{\log\cos x}{x^2}\right) &= \lim_{x\to 0}\left(\frac{\log(1-(\frac{x^2}{2!}-\frac{x^4}{4!}+..)}{x^2}\right) \\ &= \lim_{x\to 0}\left(\frac{-(\frac{x^2}{2!}-\frac{x^4}{4!}+...)-\frac{1}{2}(\frac{x^2}{2!}-\frac{x^4}{4!}+...)^2+...}{x^2}\right) \\ &= -\frac{1}{2}. \end{aligned}$$

$$(b)\ \lim_{x\to 0}\left(\frac{\log(1+x^2)}{\sin 2x}\right) = \lim_{x\to 0}\left(\frac{x^2-\frac{x^4}{2}+...}{2x-\frac{8x^3}{3!}+...}\right) = 0.$$

$$\begin{aligned} (c)\ \lim_{x\to 0}\left((\cos x)^{1/x}\right) &= \lim_{x\to 0}\left(\exp\left(\frac{\log\cos x}{x}\right)\right) = \lim_{x\to 0}\left(\exp\left(\frac{\log(1-\frac{x^2}{2!}+\frac{x^4}{4!}+...)}{x}\right)\right) \\ &= \lim_{x\to 0}\left(\exp\left(\frac{1}{x}\left(-\frac{x^2}{2}+\frac{x^4}{4!}+...-\frac{1}{2}\left(-\frac{x^2}{2}+\frac{x^4}{4!}+...\right)^2+...\right)\right)\right) \\ &= 1. \end{aligned}$$

3. From the definition of the derivative

$$
\begin{aligned}
\frac{d\sin\sqrt{x}}{dx} &= \lim_{\epsilon\to 0}\left(\frac{\sin\sqrt{x+\epsilon}-\sin\sqrt{x}}{\epsilon}\right),\\
&= \lim_{\epsilon\to 0}\left(\frac{\sin(\sqrt{x}(1+\frac{\epsilon}{x})^{1/2})-\sin\sqrt{x}}{\epsilon}\right),\\
&= \lim_{\epsilon\to 0}\left(\frac{\sin(\sqrt{x}(1+\frac{\epsilon}{2x}+...))-\sin\sqrt{x}}{\epsilon}\right),\\
&= \lim_{\epsilon\to 0}\left(\frac{\sin(\sqrt{x}+\frac{\epsilon}{2\sqrt{x}}+...))-\sin\sqrt{x}}{\epsilon}\right),\\
&= \lim_{\epsilon\to 0}\left(\frac{\sin\sqrt{x}\cos(\frac{\epsilon}{2\sqrt{x}}+...)+\cos\sqrt{x}\sin(\frac{\epsilon}{2\sqrt{x}}+...)-\sin\sqrt{x}}{\epsilon}\right),\\
&= \lim_{\epsilon\to 0}\left(\frac{\sin\sqrt{x}(1-\frac{1}{2!}(\frac{\epsilon}{2\sqrt{x}}+...)^2)+\cos\sqrt{x}(\frac{\epsilon}{2\sqrt{x}}+...)-\sin\sqrt{x}}{\epsilon}\right),\\
&= \frac{1}{2\sqrt{x}}\cos\sqrt{x}.
\end{aligned}
$$

4.(a) On use of the chain rule,

$$\frac{d(x^{10}+1)^{1/2}}{dx} = \frac{5x^9}{(x^{10}+1)^{1/2}}.$$

(b) On use of the product rule,

$$\frac{d(x\tanh x)}{dx} = \tanh x + x\mathrm{sech}^2 x$$

(c) On use of the quotient rule,

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{x}{e^x+e^{-x}}\right) &= \frac{e^x+e^{-x}-x(e^x-e^{-x})}{(e^x+e^{-x})^2}\\
&= \frac{1}{e^x+e^{-x}} - \frac{x(e^x-e^{-x})}{(e^x+e^{-x})^2}.
\end{aligned}
$$

(d) On use of the chain rule,

$$\frac{d\exp(x)^x}{dx} = \frac{d\exp(x^2)}{dx} = 2x\exp(x^2).$$

5. On use of the Leibniz rule

$$\begin{aligned}\frac{d^n x^2 e^{3x}}{dx^n} &= \sum_{j=0}^{n} \binom{n}{j} \frac{d^j x^2}{dx^j} \frac{d^{n-j} e^{3x}}{dx^{n-j}} \\ &= x^2 3^n e^{3x} + 2xn3^{n-1}e^{3x} + n(n-1)3^{n-2}e^{3x} \\ &= 3^{n-2}e^{3x}\left(9x^2 + 6xn + n(n-1)\right).\end{aligned}$$