M1M1: Progress Test 2 (2004): SOLUTIONS

1. Expanding the numerator,

$$\lim_{x\to 1}\left(\frac{x^4-1}{x-1}\right)=\lim_{x\to 1}\left(\frac{(x-1)(x^3+x^2+x+1)}{x-1}\right)=4.$$

(b) On use of the binomial series expansion,

$$\lim_{x\to 0}\left(\frac{(1+2x)^{1/2}-1}{x}\right)=\lim_{x\to 0}\left(\frac{(1+\frac{1}{2}2x-\frac{1}{4}\frac{(2x)^2}{2!}+...)-1}{x}\right)=1.$$

(c) Dividing both numerator and denominator by e^x

$$\lim_{x\to\infty}\left(\frac{2+3e^x}{1+4e^x}\right)=\lim_{x\to\infty}\left(\frac{2e^{-x}+3}{e^{-x}+4}\right)=\frac{3}{4}.$$

2.

(a) On use of the exponential series

$$\begin{aligned} e^{\sin x} &= 1+\sin x+\frac{1}{2!}\sin^2 x+... \\ &= 1+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...\right)+\frac{1}{2!}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+...\right)^2+... \\ &= 1+x+\frac{x^2}{2!}+... \end{aligned}$$

(b) On use of the sine series

$$\sin(x+x^2)=(x+x^2)-\frac{(x+x^2)^3}{3!}+...=x+x^2-\frac{x^3}{6}+...$$

(c) On use of the binomial series

$$\begin{aligned} \frac{1}{1+(1+x)^{1/2}} &= \frac{1}{1+1+\frac{x}{2}-\frac{1}{4}\frac{x^2}{2!}+...} \\ &= \frac{1}{2+\frac{x}{2}-\frac{x^2}{8}+...} \\ &= \frac{1}{2}\left(1+\frac{x}{4}-\frac{x^2}{16}+...\right)^{-1} \\ &= \frac{1}{2}\left(1-\left(\frac{x}{4}-\frac{x^2}{16}+...\right)+\left(\frac{x}{4}-\frac{x^2}{16}+...\right)^2+...\right) \\ &= \frac{1}{2}-\frac{x}{8}+\frac{x^2}{16}+... \end{aligned}$$

3. From the definition of the derivative, we need to compute

$$\frac{de^{\sqrt{x}}}{dx} = \lim_{\epsilon \to 0} \left(\frac{e^{(x+\epsilon)^{1/2}} - e^{x^{1/2}}}{\epsilon} \right).$$

$$\begin{aligned}
\lim_{\epsilon \to 0} \left(\frac{e^{(x+\epsilon)^{1/2}} - e^{x^{1/2}}}{\epsilon} \right) &= \lim_{\epsilon \to 0} \left(\frac{e^{x^{1/2}(1+\frac{\epsilon}{x})^{1/2}} - e^{x^{1/2}}}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{e^{x^{1/2}(1+\frac{\epsilon}{2x}+...)} - e^{x^{1/2}}}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{e^{x^{1/2}} e^{\left(\frac{\epsilon}{2x^{1/2}}+...\right)} - e^{x^{1/2}}}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{e^{x^{1/2}} (1 + \left(\frac{\epsilon}{2x^{1/2}} + ...\right) + \frac{1}{2!} \left(\frac{\epsilon}{2x^{1/2}} + ...\right)^2 + ... - 1)}{\epsilon} \right) \\
&= \frac{e^{x^{1/2}}}{2x^{1/2}}
\end{aligned}$$

4. By definition,

$$4^x = e^{x \log 4}.$$

Therefore

$$\frac{d4^x}{dx} = \log 4 * e^{x \log 4} = \log 4 * 4^x.$$

(b) By the chain rule,

$$\frac{d \log(1 + e^{2x})}{dx} = \frac{2e^{2x}}{1 + e^{2x}}.$$

(c) By the chain rule,

$$\frac{d \tan(x^x)}{dx} = \frac{d \tan(e^{x \log x})}{dx} = \sec^2(x^x)\,(1 + \log x)\, x^x.$$

5. By the Leibniz rule, and on use of the fact that

$$\frac{d^n e^{-2x}}{dx^n} = (-2)^n e^{-2x}, \quad n \geq 0,$$

it follows that

$$\frac{d^n f(x)}{dx^n} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j x^2}{dx^j} \frac{d^{n-j} e^{-2x}}{dx^{n-j}},$$

$$= x^2(-2)^n e^{-2x} + 2nx(-2)^{n-1}e^{-2x} + 2\frac{n(n-1)}{2!}(-2)^{n-2}e^{-2x}.$$

Therefore

$$f^n(0) = (-2)^{n-2}n(n-1).$$