M1M1: Progress Test 2 (2005): SOLUTIONS

1.(a) From the definition of the derivative

$$\begin{aligned}
\frac{d\exp(x^2)}{dx} &= \lim_{\epsilon\to 0}\left[\frac{\exp((x+\epsilon)^2)-\exp(x^2)}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{\exp(x^2+2x\epsilon+\epsilon^2)-\exp(x^2)}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{\exp(x^2)(\exp(2x\epsilon+\epsilon^2)-1)}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{\exp(x^2)(1+(2x\epsilon+\epsilon^2)+\ldots-1)}{\epsilon}\right] \\
&= 2x\exp(x^2).
\end{aligned}$$

(b) Again, from the definition of the derivative

$$\begin{aligned}
\frac{d(1+x^2)^{1/2}}{dx} &= \lim_{\epsilon\to 0}\left[\frac{(1+(x+\epsilon)^2)^{1/2}-(1+x^2)^{1/2}}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{(1+x^2+2x\epsilon+\epsilon^2)^{1/2}-(1+x^2)^{1/2}}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{(1+x^2)^{1/2}\left[\left(1+\frac{2\epsilon x+\epsilon^2}{1+x^2}\right)^{1/2}-1\right]}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[\frac{(1+x^2)^{1/2}\left[1+\frac{1}{2}\frac{2\epsilon x+\epsilon^2}{1+x^2}+\ldots-1\right]}{\epsilon}\right] \\
&= \frac{x}{(1+x^2)^{1/2}}.
\end{aligned}$$

2.(a) By the chain rule,

$$\frac{d}{dx}\left(1+\cos x\right)^{1/3} = -\frac{1}{3}\frac{\sin x}{(1+\cos x)^{2/3}}$$

(b) By definition

$$\frac{d}{dx}\left(\cos x\right)^{x} \equiv \frac{d}{dx}\exp(x\log\cos x).$$

Hence

$$\frac{d}{dx}(\cos x)^x = \exp(x \log \cos x) \frac{d}{dx}(x \log \cos x) = (\cos x)^x (\log \cos x - x \tan x).$$

(c) By the chain rule

$$\frac{d}{dx} \cos(\sin^{-1}(x)) = -\sin(\sin^{-1}(x)) \frac{d}{dx} \sin^{-1}(x) = -\frac{x}{\sqrt{1-x^2}}.$$

3. There are (at least) two methods. The first is to note that

$$\begin{aligned}\frac{x^3}{1+x} = \frac{x^3+x^2-x^2}{1+x} = x^2 - \frac{x^2}{1+x} = x^2 - \frac{x^2+x-x}{1+x} &= x^2 - x + \frac{x}{1+x} \\ &= x^2 - x + 1 - \frac{1}{1+x}.\end{aligned}$$

It follows that

$$\frac{d^{10}}{dx^{10}}\left(\frac{x^3}{1+x}\right) = -\frac{d^{10}}{dx^{10}}\left(\frac{1}{1+x}\right)$$

since the tenth derivative of a quadratic polynomial vanishes. But,

$$\frac{d}{dx}\frac{1}{1+x} = -\frac{1}{(1+x)^2}; \quad \frac{d^2}{dx^2}\frac{1}{1+x} = \frac{(-1)(-2)}{(1+x)^3}; \quad \frac{d^3}{dx^3}\frac{1}{1+x} = \frac{(-1)(-2)(-3)}{(1+x)^4}, \ldots$$

it is therefore clear that

$$\frac{d^n}{dx^n}\frac{1}{1+x} = (-1)^n \frac{n!}{(1+x)^{n+1}}. \qquad (*)$$

Hence

$$\frac{d^{10}}{dx^{10}}\left(\frac{x^3}{1+x}\right) = -\frac{10!}{(1+x)^{11}}.$$

Alternatively, using the Leibniz rule,

$$\frac{d^{10}}{dx^{10}}\left(\frac{x^3}{1+x}\right) = \sum_{j=0}^{10} \binom{10}{j} \frac{d^j}{dx^j}x^3 \frac{d^{10-j}}{dx^{10-j}}\left(\frac{1}{1+x}\right)$$

which yields

$$\frac{d^{10}}{dx^{10}}\left(\frac{x^3}{1+x}\right) = x^3\frac{d^{10}}{dx^{10}}\frac{1}{1+x} + \binom{10}{1}3x^2\frac{d^9}{dx^9}\frac{1}{1+x} + \binom{10}{2}6x\frac{d^8}{dx^8}\frac{1}{1+x} + \binom{10}{3}6\frac{d^7}{dx^7}\frac{1}{1+x}.$$

This simplifies, using the result (*), to

$$\frac{d^{10}}{dx^{10}}\left(\frac{x^3}{1+x}\right) = \frac{10!}{(1+x)^{11}}\left(x^3 - 3x^2(1+x) + 3x(1+x)^2 - (1+x)^3\right) = -\frac{10!}{(1+x)^{11}}.$$

4. Note that

$$y^2 = x^2 - x^4 = x^2(1-x^2)$$

so the curve only exists provided $|x| \le 1$.

The curve is invariant under transformations $x \mapsto -x$ and $y \mapsto -y$ hence it is reflectionally symmetric about both x and y axes.

As $x \to 0$, $y \to \pm x$. The points $(0,0), (\pm 1, 0)$ are clearly on the curve.

On differentiation,

$$2y\frac{dy}{dx} = 2x - 4x^3$$

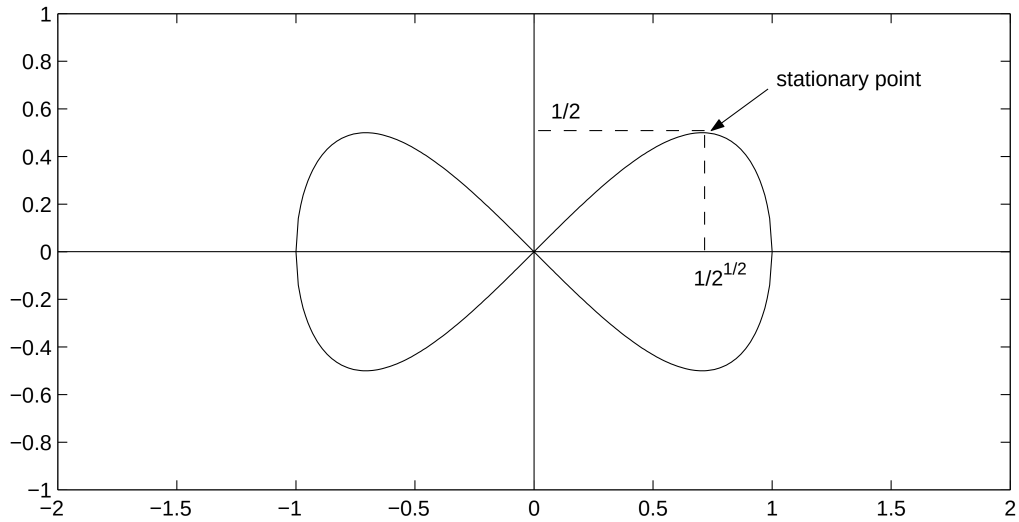
so that

$$\frac{dy}{dx} = \frac{x(2-4x^2)}{2y}$$

so there are stationary points at

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$$

while slope becomes infinite at $x = \pm 1$.


stationary point
1/2
1/2^{1/2}
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2