M1M1: Progress Test 3 (2002): SOLUTIONS

1.(i) $z^3-(3-i)z^2+(2-3i)z+2i=0.$

This can be factorized to

$$(z+i)(z-1)(z-2)=0.$$

The solutions are $z=-i, 1$ and 2.

(b) Note that

$$z^2\bar{z}^2+z\bar{z}-6=(z\bar{z}-2)(z\bar{z}+3)=0.$$

Thus, solutions are any z satisfying $|z|=\sqrt{2}$ – i.e., points on the circle centred at the origin of radius $\sqrt{2}$.

(c) We need $|z|$ to be real and positive. Therefore solutions are $|z|=(2n+1)\pi$ where $n\geq 0$ is an integer. These points lie on infinitely many circles, centred at $z=0$, with radii $\pi, 3\pi, 5\pi$.....

(d) Rewrite the equation as

$$\tanh z=\frac{e^z-e^{-z}}{e^z+e^{-z}}=2.$$

Rearranging, we get

$$e^{2z}=-3.$$

But

$$-3=e^{\log 3+i\pi+2k\pi i}$$

where k is any integer. Therefore we identify

$$2z=\log 3+i\pi+2k\pi i,$$

or

$$z=\frac{1}{2}\log 3+\frac{i\pi}{2}+k\pi i,$$

where k is any integer.

2. We know

$$e^{4i\theta}=\cos 4\theta+i\sin 4\theta.$$

But

$$\begin{aligned}e^{4i\theta}&=(\cos\theta+i\sin\theta)^4\\&=\cos^4\theta+4i\sin\theta\cos^3\theta-6\cos^2\theta\sin^2\theta-4i\sin^3\theta\cos\theta+\sin^4\theta.\end{aligned}$$

Equating real parts gives

$$\cos 4\theta=\cos^4\theta-6\sin^2\theta\cos^2\theta+\sin^4\theta.$$

But $\sin^2\theta = 1 - \cos^2\theta$, so

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Therefore $c_0 = 1;\ c_1 = 0;\ c_2 = -8;\ c_3 = 0;\ c_4 = 8.$

3. (a) Putting integrand in partial fraction form gives

$$\begin{aligned}\int_0^1 \frac{(x-1)^2}{x^2+4x+3}dx &= \int_0^1 1 - \frac{8}{x+3} + \frac{2}{x+1}dx \\ &= \Big[x - 8\log|x+3| + 2\log|x+1|\Big]_0^1 \\ &= 1 - 8\log(4/3) + 2\log 2.\end{aligned}$$

(b) Making use of the substitution $y^2 = 2 - x$,

$$\int_0^1 \sqrt{\frac{x}{2-x}}dx = \int_{\sqrt{2}}^1 -\frac{2ydy}{y}\sqrt{2-y^2} = \int_1^{\sqrt{2}} 2\sqrt{2-y^2}dy$$

Now let $y = \sqrt{2}\sin\theta$, integral becomes

$$\begin{aligned}2\int_{\pi/4}^{\pi/2} \sqrt{2}\cos^2\theta d\theta\sqrt{2} &= 4\int_{\pi/4}^{\pi/2} \frac{\cos 2\theta + 1}{2}d\theta \\ &= 4\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_{\pi/4}^{\pi/2} \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

(c) By inspection

$$\int_0^\pi \cos^4 x \sin x dx = \left[-\frac{\cos^5 x}{5}\right]_0^\pi = \frac{2}{5}.$$