M1M1: Progress Test 3 (2003): SOLUTIONS

1. On use of Taylor's theorem, for $0 \le x \le 1/2$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \frac{x^4}{4!} f^{(iv)}(\bar{x})$$

where $\bar{x}$ is between 0 and $1/2$. Direct calculation shows that

$$f^{(iv)}(x) = \frac{24}{(1+x)^5}$$

The error E incurred in the approximation is therefore

$$E = \frac{1}{4!}\frac{24x^4}{(1+\bar{x})^5} \le \frac{1}{2^4} = \frac{1}{16},$$

which gives an upper bound on the error.

2.(a) Note that

$$z^3 = -1 = e^{3\pi i/2 + 2k\pi i}$$

where k is any integer. Therefore,

$$z = e^{\pi i/2 + 2k\pi i/3}$$

where $k = 0, 1, 2$ give distinct roots.

(b) Using definition of $\tanh z$,

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = 3.$$

On rearrangement, this reduces to

$$e^{2z} = -2 = e^{\log 2 + i\pi + 2k\pi i}$$

where k is any integer. We identify the solutions as

$$z = \frac{1}{2}\left[\log 2 + i\pi + 2k\pi i\right]$$

where k is any integer.

Let

$$w = \frac{z-i}{z+i}$$

Then $\arg[w] = 0$ or π is equivalent to

$$\bar{w} = w,$$

which yields

$$\frac{\bar{z}+i}{\bar{z}-i} = \frac{z-i}{z+i}.$$

Rearrangement yields

$$z+\bar{z}=0, \quad \text{or} \quad x=0.$$

So the solutions are all points on the imaginary axis, excluding $\pm i$.

3. On use of the substitution $u = e^x$,

$$\int \frac{e^x}{e^x+2}dx = \int \frac{du}{u+2} = \log|u+2| + c = \log|e^x+2| + c$$

where c is some constant of integration.

On use of integration by parts,

$$\int xe^{2x}dx = \left[\frac{xe^{2x}}{2}\right] - \int \frac{e^{2x}}{2}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

First note that

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2 - 1/4 + 1} = \int \frac{dx}{(x+1/2)^2 + 3/4}$$

Now let $x + 1/2 = \sqrt{3}u/2$. Then

$$\int \frac{\sqrt{3}du}{2} \frac{1}{3/4(u^2+1)} = \frac{2}{\sqrt{3}} \tan^{-1} u + c = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + c.$$

4. On use of integration by parts

$$\begin{aligned} I_n = \int_0^1 x^n \log x dx &= \left[\frac{x^{n+1}}{n+1}\log x\right]_0^1 - \int_0^1 \frac{x^n}{n+1}dx \\ &= -\frac{1}{n+1}\int_0^1 x^n dx \\ &= -\frac{1}{(n+1)^2}. \end{aligned}$$

Therefore

$$I_{100} = -\frac{1}{101^2}.$$

Again, using integration by parts,

$$
\begin{aligned}
J_n = \int_0^1 x(\log x)^n dx &= \left[\frac{x^2}{2}(\log x)^n\right]_0^1 - \int_0^1 \frac{x^2}{2} n(\log x)^{n-1}\frac{dx}{x} \\
&= -\frac{n}{2}\int_0^1 x(\log x)^{n-1} dx \\
&= -\frac{n}{2} J_{n-1}.
\end{aligned}
$$

Therefore

$$J_{100} = \left(-\frac{1}{2}\right)^{100} 100!\ J_0.$$

But

$$J_0 = \int_0^1 x dx = \frac{1}{2},$$

so

$$J_{100} = \frac{100!}{2^{101}}.$$