M1M1: Progress Test 3 (2004): SOLUTIONS

1. First, write

$$y(x) = \frac{2(x^2 - x) + 2x + 1}{x - 1} = 2x + \frac{2(x-1)+3}{x-1} = 2x + 2 + \frac{3}{x-1}.$$

When $x = 0$, $y = -1$.

There is a vertical asymptote at $x = 1$.

As $x \to \pm\infty$, $y \to 2x + 2$.

Taking a derivative (to find stationary points),

$$\frac{dy}{dx} = 2 - \frac{3}{(x-1)^2}$$

there are stationary points at $x = 1 \pm \sqrt{3/2}$.

2. Applying the mean value theorem to the function $\log x$, we have

$$\frac{\log b - \log a}{b - a} = \frac{1}{c}$$

for some c between a and b. Taking $a = 1$ and $b = 2$ yields

$$\log 2 = \frac{1}{c}$$

for some c between 1 and 2. But this can be used to deduce that

$$\frac{1}{2} < \log 2 < 1.$$

Alternatively: It is known that e is between 2 and 3 (in fact, $e = 2.718...$) so one can deduce, from the monotonic-increasing property of $\log x$, that

$$\log 2 < \log e = 1.$$

On the other hand,

$$\log 2 = -\log\left(\frac{1}{2}\right)$$

so use of the expansion

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ...$$

with $x = 1/2$ leads to result that

$$\log 2 = \frac{1}{2} + \text{positive terms.}$$

So

$$\frac{1}{2} < \log 2.$$

3. By Taylor's theorem,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5!}\frac{d^5\log(1+x)}{dx^5}\bigg|_{\bar{x}}$$

for some $\bar{x}$ between 0 and x. Direct calculation of derivatives reveals that

$$\frac{d^5\log(1+x)}{dx^5}\bigg|_{\bar{x}} = \frac{24}{(1+\bar{x})^5}.$$

The error E is therefore

$$E = \frac{x^5}{5!}\frac{24}{(1+\bar{x})^5}.$$

Thus,

$$|E| \leq \frac{32}{5*32} = \frac{1}{5},$$

where we have used the facts that

$$|x| < \frac{1}{2}$$

and

$$\frac{1}{(1+\bar{x})^5} \leq 2^5 \quad \text{for} \quad |\bar{x}| \leq 1/2.$$

(attained when $\bar{x} = -\frac{1}{2}$).

4. On use of the definition of $\tanh z$, equation becomes

$$\frac{e^z - e^{-z}}{e^z + e^{-z}} = 2.$$

Rearranging

$$e^{2z} = -3 = e^{\log 3 + i\pi + 2k\pi i}$$

where k is any integer. We therefore identify

$$z = \frac{1}{2}\log 3 + \frac{i\pi}{2} + k\pi i$$

where k is any integer.

(b) By the previous result we would need

$$|z| = \frac{1}{2}\log 3 + \frac{i\pi}{2} + k\pi i.$$

But $|z|$ must be **real**, so there are no solutions to the equation.

(c) Multiplying by $\bar{z}$,

$$z\bar{z} + 1 = 2\bar{z}.$$

But using $z = x + iy$,

$$x^2 + y^2 + 1 = 2x - 2iy.$$

Equating real and imaginary parts of both sides means we must have $x = 1, y = 0$ so $z = 1$ is the only solution.

5. (a) Note that

$$\int \frac{x-1+2}{x-1}dx = \int 1 + \frac{2}{x-1}dx = x + 2\log|x-1| + c$$

where c is a constant. The answer

$$x + 2\log(x-1) + c.$$

is equally acceptable.

(b) Rewriting the integrand gives

$$\int \frac{x^2-1+2}{x^2-1}dx = \int 1 + \frac{2}{x^2-1}dx = \int 1 + \frac{1}{x-1} - \frac{1}{x+1}dx$$
$$= x + \log|x-1| - \log|x+1| + c.$$

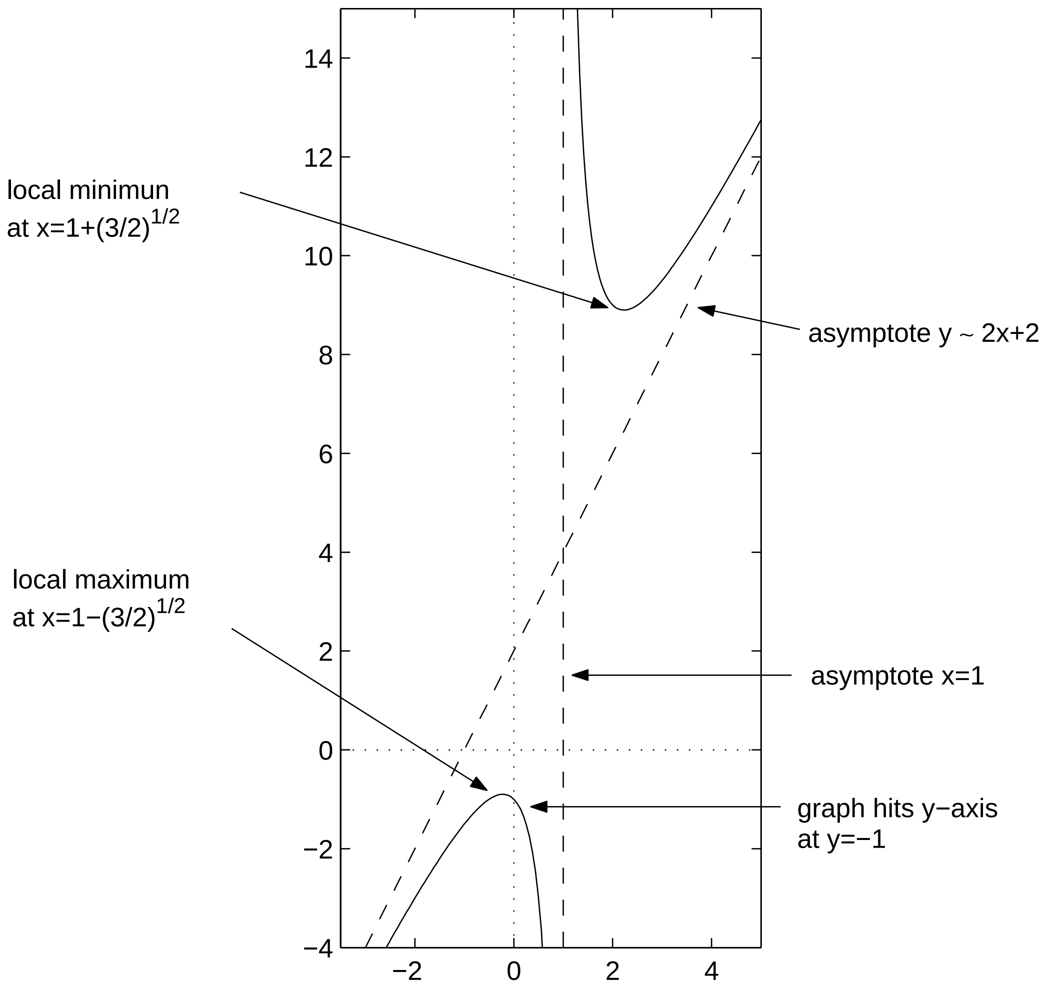
The answer

$$x + \log(x-1) - \log(x+1) + c.$$

is equally acceptable.

(c) Again, rewriting the integrand gives

$$\int \frac{x^2-1}{x^2+1}\, dx = \int 1 - \frac{2}{x^2+1}\, dx = x - 2\tan^{-1} x + c.$$


local minimun
at x=1+(3/2)^1/2
local maximum
at x=1−(3/2)^1/2
asymptote y ~ 2x+2
asymptote x=1
graph hits y−axis
at y=−1
14
12
10
8
6
4
2
0
−2
−4
−2
0
2
4