M1M1: Progress Test 3 (2005): SOLUTIONS

1.(a) On use of the well-known exponential series, we have

$$f(x) = x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...\right) = x - x^2 + \frac{x^3}{2} + ...$$

(b) By Taylor's theorem, the error incurred in using this approximation in the interval $0 \leq x \leq 1$ is

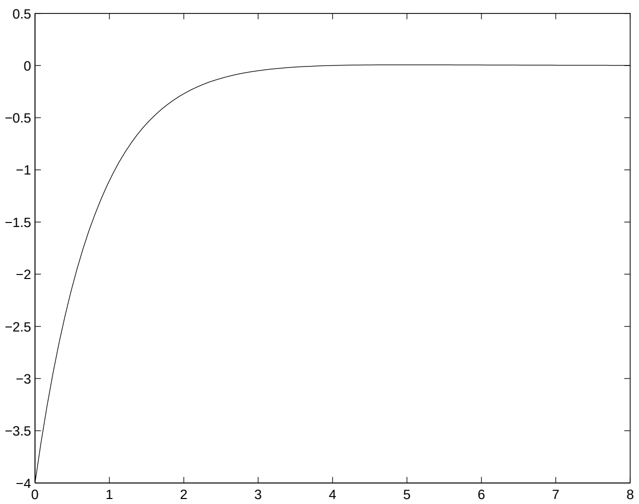
$$E = \frac{x^4}{4!} f^{(iv)}(\bar{x})$$

for some $0 \leq \bar{x} \leq 1$ and where $f^{(iv)}$ denotes the fourth derivative of $f(x)$. On repeated differentiation,

$$f'(x) = (1-x)e^{-x},$$
$$f''(x) = (x-2)e^{-x},$$
$$f'''(x) = (3-x)e^{-x},$$
$$f^{(iv)}(x) = (x-4)e^{-x},$$
$$f^{(v)}(x) = (5-x)e^{-x}, ...$$

It is helpful to consider a graph of $f^{(iv)}(x)$. It goes through the point $(0, -4)$ and tends to zero as $x \to \infty$ and there is a stationary point at $x = 5$, i.e., at the point $(5, e^{-5})$. Thus, over the interval $0 \leq x \leq 1$ the maximum value of $|f^{(iv)}(x)|$ is 4 (attained when $x = 0$). Thus

$$|E| \leq \frac{1}{4!} 4 = \frac{1}{6}.$$



2. (a) Notice, by inspection, that $z = i$ is a solution. Factorizing yields

$$(z - i)(z^2 + z + 1) = 0$$

thus, on use of the formula for solutions of a quadratic, the three solutions are

$$z = i, \quad \frac{-1 \pm i\sqrt{3}}{2}.$$

(b) By definition we have

$$\frac{e^z + e^{-z}}{e^z - e^{-z}} = 2$$

which, on rearrangement, becomes

$$e^{2z} = 3.$$

But this can be written

$$e^{2z} = e^{\log 3 + 2k\pi i}$$

where k is any integer. Therefore, the solutions are given by

$$z = \frac{\log 3}{2} + k\pi i$$

where k is any integer.

3. On rearrangement, notice that

$$(\zeta - 2)z = \zeta + i$$

or

$$\zeta = \frac{i + 2z}{z - 1}$$

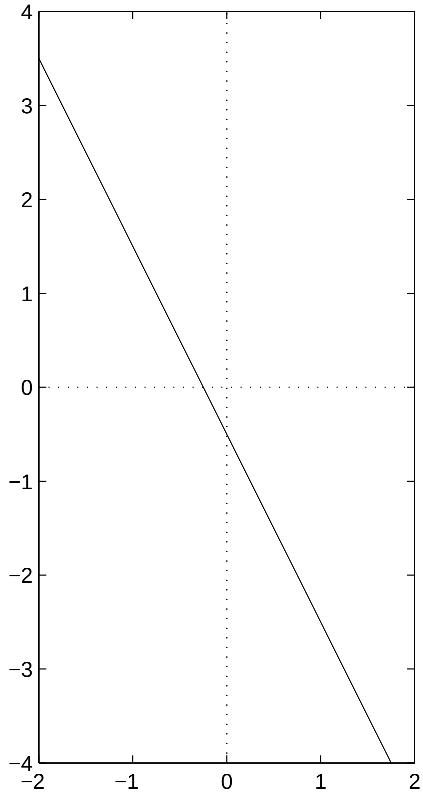
Substituting this into

$$(\zeta - 1)(\bar{\zeta} - 1) = 1$$

and simplifying, with $z = x + iy$, yields

$$4x + 2y = -1$$

which is a straight line (see graph).



4. (a) Let $x = \sqrt{2}y$ then integral becomes

$$\int \frac{\sqrt{2}dy}{2(y^2+1)} = \frac{1}{\sqrt{2}} \tan^{-1} y + \text{constant} = \frac{1}{\sqrt{2}} \tan^{-1}(x/\sqrt{2}) + \text{constant}.$$

(b) Using partial fractions,

$$\begin{aligned} \int \frac{dx}{x^2-2} &= \int \frac{1}{2\sqrt{2}} \left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}} \right) \\ &= \frac{1}{2\sqrt{2}} \left(\log(x-\sqrt{2}) - \log(x+\sqrt{2}) \right) + \text{constant}. \end{aligned}$$

(c) Using partial fractions,

$$\begin{aligned} \int \frac{dx}{x^3-x^2+x-1} &= \int \frac{1}{2} \left(\frac{1}{x-1} - \frac{(x+1)}{x^2+1} \right) dx \\ &= \frac{1}{2} \log(x-1) - \frac{1}{2} \tan^{-1} x - \frac{1}{4} \log(1+x^2) + \text{constant}. \end{aligned}$$