M1M1: Problem Sheet 1

Functions

1. Solve the following equations:

(a)
$$x^2 - 4x - 4 = 0;$$

(b)
$$5x^2 - 8x + 3 = 0;$$

(c)
$$\frac{2u+3}{u-1} = \frac{3u-2}{5u-21};$$

(d)
$$1 = \frac{10}{v-2} + \frac{6}{1-v}.$$

2.(a) If
$$f(x) = \frac{x+1}{x-1} \quad \text{and} \quad g(x) = \frac{x-1}{x+1},$$
find expressions for the functions $f(g(x)), g(f(x)), f(f(x))$ and $g(g(x))$.
(b) Find the function inverse to $f(x) = \frac{x-2}{x-1}$. What does your result tell you about the symmetry of $f(x)$?

3. The *Heaviside step function* $H(x)$ is defined as follows:
$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(i) Sketch graphs of the functions $H(x-1)$ and $H(x) - H(x-1)$;

(ii) The function $f(x)$ is defined as
$$f(x) = \begin{cases} x, & x \geq 0, \\ x^2, & x < 0. \end{cases}$$
Find a single expression for $f(x)$ in terms of the functions x, x^2 and $H(x)$.

4. It is known that any real function can be decomposed into the sum of an even and an odd function. Decompose the following functions in this way:

$$(a)\ \frac{1}{x+1}; \quad (b)\ \left(\frac{1-x}{1+x}\right)^{1/2}; \quad (c)\ \sin(x+1).$$

5. A function $f(x)$ has the form

$$f(x) = \frac{ax+b}{cx+d}$$

where a, b, c and d are real constants.

(a) Find the inverse function $f^{-1}(x)$;

(b) Find the relation between the constants a, b, c and d such that $f^{-1}(x) = f(x)$. Verify your result by computing the composed function $f(f(x))$;

(c) If a new function $g(x) = f(x-e)$ where $e = (a+d)/c$, use the result of part (a) to show that $g^{-1}(x) = g(x)$.

6. The exponential function $\exp(x)$ is defined by the infinite series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which is convergent for all values of x. By multiplying together the two series expansions for $\exp(x)$ and $\exp(y)$ and collecting terms of the same degree (e.g. x^3, x^2y, xy^2 and y^3 are all of degree 3), verify that

$$\exp(x)\exp(y) = \exp(x+y).$$

7. The exponential function $y = \exp(x)$ defined in Q6 has an inverse function which is denoted $y = \log x$. By using the properties of the exponential function established in Q6, establish the following results concerning its inverse function:

$$\log(uv) = \log u + \log v; \quad \log\left(\frac{1}{u}\right) = -\log u; \quad \log\left(\frac{u}{v}\right) = \log u - \log v.$$

8. Find all solutions x to the following two equations:

$$(a)\ 4\sin^2 x - 5\cos x - 5 = 0; \qquad (b)\ 2\sec^2 x - \tan x - 3 = 0.$$

9. Use the identities

$$\sin(x+y) = \sin x \cos y + \sin y \cos x,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

to derive expressions for $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin\alpha + \sin\beta$, $\cos\alpha + \cos\beta$.

10. Given that $t = \tan\frac{x}{2}$, show that

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}.$$