M1M1: Problem Sheet 1: SOLUTIONS

Functions

1. Using formula for roots of a quadratic

$$(a)\ x = \frac{4 \pm \sqrt{16+16}}{2} = 2 \pm \sqrt{8}; \quad (b)\ x = \frac{8 \pm \sqrt{64-60}}{10} = 1, \frac{3}{5};$$

$$(c)\ (2u+3)(5u-21) = (3u-2)(u-1)$$
$$7u^2 - 22u - 65 = 0$$
$$u = 5, -\frac{13}{7}.$$

$$(d)\ (v-2)(1-v) = 10(1-v) + 6(v-2)$$
$$v(v-7) = 0$$
$$u = 0, 7.$$

2.(a)

$$f(g(x)) = \frac{\frac{x-1}{x+1}+1}{\frac{x-1}{x+1}-1} = \frac{x-1+(x+1)}{x-1-(x+1)} = -x.$$
$$g(f(x)) = \frac{\frac{x+1}{x-1}-1}{\frac{x+1}{x-1}+1} = \frac{x+1-(x-1)}{x+1+(x-1)} = \frac{1}{x}.$$
$$f(f(x)) = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{x+1+(x-1)}{x+1-(x-1)} = x.$$
$$g(g(x)) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{x-1-(x+1)}{x-1+(x+1)} = -\frac{1}{x}.$$

(b) Let

$$y = \frac{x-2}{x-1}$$

Then, rearranging,

$$x = \frac{y-2}{y-1}$$

so that

$$f^{-1}(x) = \frac{x-2}{x-1} = f(x)$$

The function $f(x)$ has a graph that is reflectionally-symmetric about the line $y = x$.

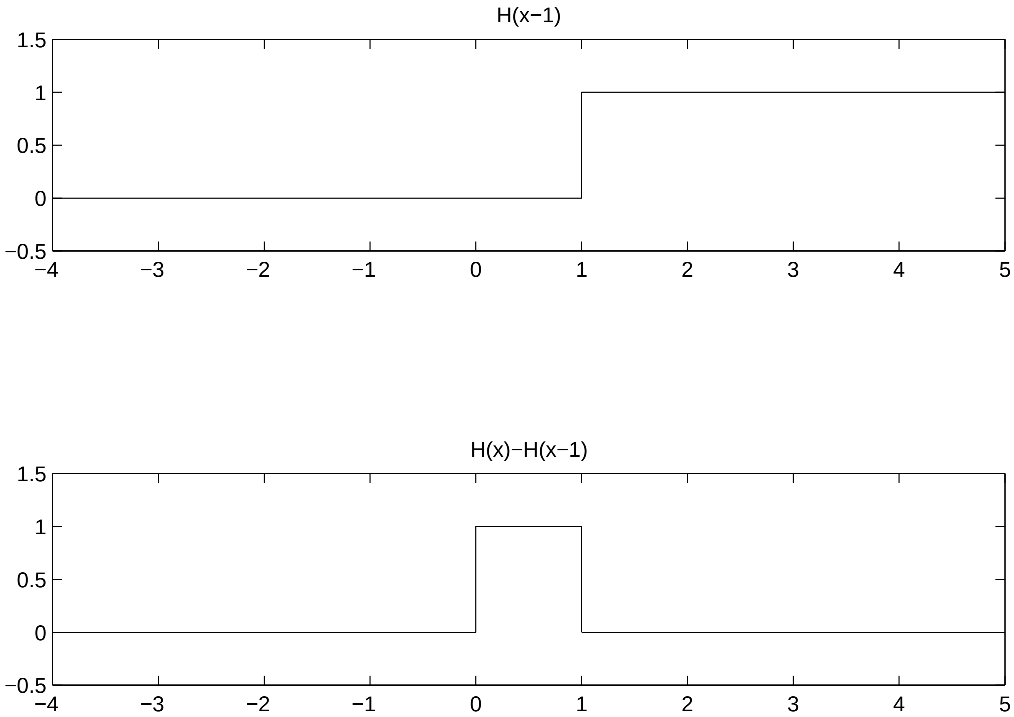

Figure 1: Graphs for question 3(a)

3. (a) See Figure 1.

(b) $f(x) = x^2 + H(x)(x - x^2)$

4.

$$(a)\ \frac{1}{x+1} = \frac{1}{x+1}\left(\frac{x-1}{x-1}\right) = \frac{x-1}{x^2-1} = \frac{x}{x^2-1} - \frac{1}{x^2-1}$$

$$(b)\ \left(\frac{1-x}{1+x}\right)^{1/2} = \left(\frac{1-x}{1+x}\right)^{1/2}\left(\frac{1-x}{1-x}\right)^{1/2} = \frac{1-x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$(c)\ \sin(x+1) = \sin x \cos 1 + \sin 1 \cos x.$$

5. Let

$$y = \frac{ax+b}{cx+d}$$

Rearranging,

$$x = \frac{-dy + b}{cy - q}$$

So

$$f^{-1}(x) = \frac{-dx + b}{cx - q}$$

By inspection, $f(x) = f^{-1}(x)$ if and only if $a = -d$.

By direct calculation, can check $f(f(x)) = x$.

$$g(x) = \frac{a(x-e) + b}{c(x-e) + d} = \frac{ax - ea + b}{cx - a - d + d} = \frac{ax - ea + b}{cx - a}$$

where we have substituted the value of e. This expressions clearly has "$d = -a$" as required for $g^{-1}(x) = g(x)$.

6.

$$\begin{aligned}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + ...\right) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \\ &+ y + xy + \frac{x^2 y}{2!} + ... \\ &+ \frac{y^2}{2!} + \frac{xy^2}{2!} + ... \\ &+ \frac{y^3}{3!} + ... \\ &= 1 + (x+y) + \frac{(x+y)^2}{2!} \\ &+ \frac{(x+y)^3}{3!} + ... \\ &= \exp(x+y).\end{aligned}$$

7. We have, for all $u > 0$ and $v > 0$,

$$u = \exp(\log(u)); v = \exp(\log(v)); uv = \exp(\log(uv)).$$

But

$$\exp(\log(uv)) = uv = \exp(\log(u))\exp(\log(v)) = \exp(\log(u) + \log(v))$$

where the last equality follows from Q6. We deduce that

$$\log(uv) = \log(u) + \log(v).$$

Now

$$\exp(0) = 1 = u\frac{1}{u} = \exp(\log(u))\exp(\log(u^{-1})) = \exp(\log(u) + \log(u^{-1}))$$

where the last equality follows from Q6. Hence

$$0 = \log(u) + \log(u^{-1}) \quad \text{or} \quad \log(u^{-1}) = -\log(u)$$

Finally, combining the two results above gives

$$\log(u/v) = \log(u) + \log(1/v) = \log(u) - \log(v).$$

8.(a) On use of $\sin^2 x = 1 - \cos^2 x$,

$$4\cos^2 x + 5\cos x + 1 = 0$$

which has solutions

$$\cos x = -1, -\frac{1}{4}.$$

Hence

$$x = (2n+1)\pi, \pm\cos^{-1}(-1/4) + 2n\pi$$

where n is any integer and $\cos^{-1} x$ is assumed to take a value between 0 and π.

(b) On use of $\sec^2 x = 1 + \tan^2 x$,

$$2\tan^2 x - \tan x - 1 = 0$$

Hence

$$\tan x = 1, -\frac{1}{2}.$$

Thus,

$$x = (n + 1/4)\pi, n\pi - \tan^{-1}(1/2)$$

where n is any integer and $\tan^{-1} x$ is assumed to take a value between 0 and π.

9. Letting $x = y = \theta$,

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Now let $x + y = \alpha,\ x - y = \beta$, then

$$x = \frac{\alpha+\beta}{2}, \quad y = \frac{\alpha-\beta}{2},$$

Then

$$\sin(\alpha) = \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
$$\sin(\beta) = \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Adding these two formulas gives

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Similarly,

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

10. Using the results of Q9,

$$\sin x = 2\sin(x/2)\cos(x/2) = 2\frac{\tan(x/2)}{\sec^2(x/2)} = \frac{2t}{1+t^2}.$$

Similarly,

$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \frac{1}{\sec^2(x/2)} - \frac{\tan^2(x/2)}{\sec^2(x/2)} = \frac{1-t^2}{1+t^2}.$$