M1M1: Problem Sheet 2

Series expansions and limits

1. The hyperbolic sine and cosine, denoted $\sinh x$ and $\cosh x$, are respectively the odd and even parts of the exponential function $\exp(x)$. Use this fact to find the series expansions for $\sinh x$ and $\cosh x$.

2. The hyperbolic tangent, $\tanh x$, is defined as

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

Use the series expansions derived in Q1 to find the first 3 non-zero terms in the series expansion of $\tanh x$.

3. Derive the following expression for the inverse hyperbolic tangent

$$\tanh^{-1} x = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right).$$

Use this expression to find the series expansion of $\tanh^{-1} x$.

4. Find the first three non-zero terms in the series expansions of the following functions:

(a) $(1+x)\exp(x)$; (b) $\sin(x+1)$; (c) $\exp(x)\log(1+x)$;

(d) $\dfrac{1}{2-\exp(x)}$; (e) $\sec x$; (f) $\tan x$;

(g) $\log(1+\exp(x))$; (h) $\cos(\sin(x))$.

5. Evaluate the following limits:

(a) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x}$; (b) $\lim\limits_{x\to -2} \dfrac{x^2+5x+6}{x^2+3x+2}$; (c) $\lim\limits_{x\to\infty} \dfrac{x^5+7x^3}{4x^5+x^2}$

(d) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$; (e) $\lim\limits_{x\to\infty} \dfrac{(1+x^2)^{1/2}}{x}$; (f) $\lim\limits_{x\to\pi} \dfrac{1+\cos x}{\tan^2 x}$;

(g) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$; (h) $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^2-5x+4}$; (i) $\lim\limits_{x\to 2} \dfrac{\tan(p(x-2))}{\tan(q(x-2))}$

6. Use the result that $\lim_{x\to\infty} xe^{-\alpha x} = 0$ for $\alpha > 0$ to show that

$$\lim_{t\to 0^+} t^\alpha \log t = 0$$

where the notation $t \to 0^+$ means that t tends to zero through positive values.

7. Evaluate the following limits:

$$(a)\ \lim_{x\to 0} \frac{x+\sin x}{x+x^2};\quad (b)\ \lim_{x\to 1} \frac{\log x}{x^2-1};\quad (c)\ \lim_{x\to \pi/2} \frac{1-\sin x}{(x-\pi/2)^2};$$

$$(d)\ \lim_{x\to \pi/2} (\sec x - \tan x);\quad (e)\ \lim_{x\to 0} (\sec x)^{x^{-2}};\quad (f)\ \lim_{x\to\infty} x^{1/3}\left((x+1)^{2/3} - x^{2/3}\right);$$

$$(g)\ \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right);\quad (h)\ \lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x;\quad (i)\ \lim_{x\to 1} \frac{x^2-2x+1}{x^4+x^3-7x+5}$$