
M1M1: Problem Sheet 2: SOLUTIONS

Series expansions and limits

1. The series expansion for $\exp(x)$ is

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

Collecting the even powers gives the following series expansion of $\cosh(x)$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

while, the odd terms give

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

2.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\sinh(x)}{1 + X}$$

where

$$X = \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Therefore, using the geometric series,

$$\begin{aligned} \tanh(x) &= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) (1 - X + X^2 - X^3 + X^4 + \dots) \\ &= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \left(1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} - \dots + \frac{x^4}{4} + \frac{x^6}{4!} + \dots - \frac{x^6}{8} + \dots \right) \\ &= x - \frac{x^3}{3} + \frac{2x^5}{15} + \dots \end{aligned}$$

3. Let

$$y = \tanh^{-1}(x)$$

so that

$$\tanh(y) = x.$$

Then, by definition of $\tanh(y)$,

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x, \quad \text{or,} \quad \frac{e^{2y} - 1}{e^{2y} + 1} = x.$$

Rearranging,

$$e^{2y} = \frac{1+x}{1-x}$$

so that

$$y = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right).$$

But

$$\begin{aligned} \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ \log(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

so that

$$y = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right) = \frac{1}{2} \left(\log(1+x) - \log(1-x) \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

4.

$$\begin{aligned} (a) \quad (1+x)\exp(x) &= (1+x)\left(1+x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \\ &= 1+x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + x + x^2 + \dots \\ &= 1+2x + \frac{3x^2}{2} + \dots \end{aligned}$$

$$\begin{aligned} (b) \quad \sin(x+1) &= \cos(1)\sin(x) + \sin(1)\cos(x) \\ &= \cos(1)\left(x - \frac{x^3}{3!} + \dots\right) + \sin(1)\left(1 - \frac{x^2}{2!} + \dots\right) \\ &= \sin(1) + \cos(1)x - \frac{\sin(1)}{2}x^2 + \dots \end{aligned}$$

$$\begin{aligned} (c) \quad \exp(x)\log(1+x) &= \left(1+x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right) \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \end{aligned}$$

$$(d) \frac{1}{2 - \exp(x)} = \frac{1}{2 - (1 + X)}$$

where

$$X = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Therefore

$$\begin{aligned} \frac{1}{2 - \exp(x)} &= \frac{1}{1 - X} \\ &= 1 + X + X^2 + X^3 + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + x^2 + x^3 + \dots \\ &= 1 + x + \frac{3x^2}{2} + \dots \end{aligned}$$

$$(e) \sec(x) = \frac{1}{\cos(x)} = \frac{1}{1 - X}$$

where

$$X = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots$$

Therefore

$$\begin{aligned} \sec(x) &= \frac{1}{1 - X} \\ &= 1 + X + X^2 + X^3 + \dots \\ &= 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots + \frac{x^4}{4} + \dots \\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots \end{aligned}$$

$$\begin{aligned} (f) \tan(x) &= \sin(x) \sec(x) \\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots \right) \quad \text{from (e)} \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \end{aligned}$$

$$(g) \log(1 + \exp(x)) = \log(1 + 1 + X)$$

where

$$X = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Therefore

$$\begin{aligned}\log(1 + \exp(x)) &= \log(2 + X) \\ &= \log\left(2\left(1 + \frac{X}{2}\right)\right) \\ &= \log 2 + \log\left(1 + \frac{X}{2}\right) \\ &= \log 2 + \frac{X}{2} - \frac{1}{2}\left(\frac{X}{2}\right)^2 + \frac{1}{3}\left(\frac{X}{2}\right)^3 + \dots \\ &= \log 2 + \frac{1}{2}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \frac{1}{8}\left(x^2 + \dots\right) + \dots \\ &= \log 2 + \frac{x}{2} + \frac{x^2}{8} + \dots\end{aligned}$$

$$\begin{aligned}(h) \quad \cos(\sin(x)) &= 1 - \frac{\sin^2(x)}{2!} + \frac{\sin^4(x)}{4!} + \dots \\ &= 1 - \frac{1}{2}\left(x - \frac{x^3}{3!} + \dots\right)^2 + \frac{1}{24}\left(x - \frac{x^3}{3!} + \dots\right)^4 + \dots \\ &= 1 - \frac{x^2}{2} + \frac{5x^4}{24} + \dots\end{aligned}$$

5.

$$(a) \quad \lim_{x \rightarrow 1} \left(\frac{x^3 - 1}{x}\right) = 0$$

$$(b) \quad \lim_{x \rightarrow -2} \left(\frac{(x+2)(x+3)}{(x+2)(x+1)}\right) = -1$$

$$(c) \quad \lim_{x \rightarrow \infty} \left(\frac{x^5 + 7x^3}{4x^5 + x^2}\right) = \lim_{x \rightarrow \infty} \left(\frac{x^5(1 + 7/x^2)}{x^5(4 + 1/x^3)}\right) = 1/4$$

$$(d) \quad \lim_{x \rightarrow 1} \left(\frac{x^3 - 1}{x - 1}\right) = \lim_{x \rightarrow 1} \left(\frac{(x-1)(x^2 + x + 1)}{x - 1}\right) = 3$$

$$(e) \quad \lim_{x \rightarrow \infty} \left(\frac{(1 + x^2)^{1/2}}{x}\right) = 1.$$

(f) Let $x = \pi + \epsilon$. Then limit becomes

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left(\frac{1 + \cos(\pi + \epsilon)}{\tan^2(\pi + \epsilon)} \right) &= \lim_{\epsilon \rightarrow 0} \left(\frac{1 + \cos(\pi) \cos(\epsilon) - \sin(\pi) \sin(\epsilon)}{\tan^2(\epsilon)} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \left(1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} + \dots \right)}{\left(\epsilon + \frac{\epsilon^3}{3} + \dots \right)^2} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon^2 \left(\frac{1}{2} - \frac{\epsilon^2}{4!} + \dots \right)}{\epsilon^2 \left(1 + \frac{\epsilon^2}{3} + \dots \right)^2} \right) \\ &= \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} (g) \quad \lim_{x \rightarrow 0} \left(\frac{\tan(x)}{x} \right) &= \lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \frac{1}{\cos(x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x - \frac{x^3}{3!} + \dots}{x} \frac{1}{\cos(x)} \right) \\ &= 1. \end{aligned}$$

(h) Let $x = 1 + \epsilon$. Then the limit becomes

$$\begin{aligned} \lim_{x \rightarrow 1} \left(\frac{\sin(x-1)}{(x-1)(x-4)} \right) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\sin(\epsilon)}{\epsilon(-3+\epsilon)} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon \left(1 - \frac{\epsilon^2}{3!} + \dots \right)}{\epsilon(-3+\epsilon)} \right) \\ &= -\frac{1}{3}. \end{aligned}$$

(i) Let $x = 2 + \epsilon$. Then the limit becomes

$$\lim_{\epsilon \rightarrow 0} \left(\frac{\tan(p\epsilon)}{\tan(q\epsilon)} \right) = \lim_{\epsilon \rightarrow 0} \left(\frac{p\epsilon + \frac{p\epsilon^3}{3} + \dots}{q\epsilon + \frac{q\epsilon^3}{3} + \dots} \right) = \frac{p}{q}.$$

6. Let $x = \log(1/t)$. Then $\exp(x) = 1/t$, so $\exp(-\alpha x) = t^\alpha$. Therefore

$$\lim_{x \rightarrow \infty} (x \exp(-\alpha x)) = \lim_{t \rightarrow 0^+} (-\log(t) t^\alpha) = 0$$

7.

$$(a) \lim_{x \rightarrow 0} \left(\frac{x + \left(x - \frac{x^3}{3!} + \dots \right)}{x + x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{x \left(2 - \frac{x^2}{3!} + \dots \right)}{x(1+x)} \right) = 2.$$

(b) Let $x = 1 + \epsilon$. Then

$$\begin{aligned} \lim_{x \rightarrow 1} \left(\frac{\log(x)}{(x^2 - 1)} \right) &= \lim_{\epsilon \rightarrow 0} \left(\frac{\log(1 + \epsilon)}{\epsilon(2 + \epsilon)} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon - \frac{\epsilon^2}{2} + \dots}{\epsilon(2 + \epsilon)} \right) \\ &= \frac{1}{2}. \end{aligned}$$

(c) Let $x = \pi/2 + \epsilon$.

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \sin(\pi/2 + \epsilon)}{\epsilon^2} \right) &= \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \cos(\epsilon)}{\epsilon^2} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \left(1 - \frac{\epsilon^2}{2} + \dots \right)}{\epsilon^2} \right) \\ &= \frac{1}{2}. \end{aligned}$$

(d) Let $x = \pi/2 + \epsilon$.

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \sin(\pi/2 + \epsilon)}{\cos(\pi/2 + \epsilon)} \right) &= \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \cos(\epsilon)}{-\sin(\epsilon)} \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{1 - \left(1 - \frac{\epsilon^2}{2!} + \dots \right)}{-\left(\epsilon - \frac{\epsilon^3}{3!} + \dots \right)} \right) \\ &= 0. \end{aligned}$$

$$(e) \lim_{x \rightarrow 0} \left((\sec(x))^{1/x^2} \right) = \lim_{x \rightarrow 0} \left(\exp \left(\frac{\log(\sec(x))}{x^2} \right) \right)$$

Consider

$$\lim_{x \rightarrow 0} \left(\frac{\log(\sec(x))}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{x^2} \log \left(\frac{1}{1 - X} \right) \right) = \lim_{x \rightarrow 0} \left(-\frac{1}{x^2} \log(1 - X) \right)$$

where

$$X = \frac{x^2}{2} - \frac{x^4}{4!} + \dots$$

Therefore

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{\log(\sec(x))}{x^2} \right) &= \lim_{x \rightarrow 0} \left(\frac{1}{x^2} \left(X + \frac{X^2}{2} + \frac{X^3}{3} + \dots \right) \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{x^2} \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots + \frac{x^4}{4} + \dots \right) \right) = \frac{1}{2}. \end{aligned}$$

Therefore

$$\lim_{x \rightarrow 0} \left((\sec(x))^{1/x^2} \right) = e^{1/2}.$$

$$\begin{aligned} (f) \quad \lim_{x \rightarrow \infty} \left(x^{1/3} \left((x+1)^{2/3} - x^{2/3} \right) \right) &= \lim_{x \rightarrow \infty} \left(x \left((1 + 1/x)^{2/3} - 1 \right) \right) \\ &= \lim_{x \rightarrow \infty} \left(x \left(\left(1 + \frac{2}{3x} + \binom{2}{3} \left(-\frac{1}{3} \right) \frac{1}{2!x^2} \right) - 1 \right) \right) \\ &= \frac{2}{3}. \end{aligned}$$

$$\begin{aligned} (g) \quad \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) &= \lim_{x \rightarrow 0} \left(\frac{\sin(x) - x}{x \sin(x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{x - \frac{x^3}{3!} + \dots - x}{x^2 \left(1 - \frac{x^2}{3!} + \dots \right)} \right) \\ &= 0 \end{aligned}$$

$$(h) \quad \lim_{x \rightarrow \infty} \left(1 + \frac{c}{x} \right)^x = \lim_{x \rightarrow \infty} \left(\exp(x \log(1 + c/x)) \right)$$

Now consider

$$\lim_{x \rightarrow \infty} \left(x \log(1 + c/x) \right) = \lim_{x \rightarrow \infty} \left(x \left(\frac{c}{x} - \frac{1}{2} \left(\frac{c}{x} \right)^2 + \dots \right) \right) = c.$$

Therefore,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{c}{x} \right)^x = e^c.$$

$$(i) \quad \lim_{x \rightarrow 1} \left(\frac{(x-1)^2}{(x-1)(x^3 + 2x^2 + 2x - 5)} \right) = \lim_{x \rightarrow 1} \left(\frac{(x-1)^2}{(x-1)^2(x^2 + 3x + 5)} \right) = \frac{1}{9}.$$