
M1M1: Problem Sheet 3

Differentiation

1. Find the derivatives of the following function *from first principles* (i.e., use the definition of the derivative):

(a) x^3 ; (b) $x^{1/3}$; (c) $\sqrt{x^2 - 1}$; (d) $\cos x$; (e) $\tan x$; (f) $\sin \sqrt{x}$.

2. Using any rules of differentiation that you like (e.g. the product rule, quotient rule, chain rule), find the derivatives of the following functions:

(a) $\sin(x^2)$; (b) $\sin^2 x$; (c) $\sin(2x)$; (d) 10^x ; (e) $(\sin x)^x$; (f) $\tan(\sin x)$;

(g) $\cot^{-1}(x)$; (h) $\exp(3x^2 + 5x + 2)$; (i) $\exp(-x)\cosh(2x)$; (j) $\log(x)\exp(x)$;

(k) $\sin^{-1}(x)$; (l) $\sec x$; (m) $\log|\sec x + \tan x|$; (n) $(x^2 - 1)^{1/2}$;

(o) $\frac{x}{(x^2 - 1)^{1/2}}$; (p) $\frac{1}{(x^2 - 1)^{3/2}}$; (q) $\log(x)\sin^{-1}(x)$; (r) $\tan^{-1}(x)$.

3. A curve in the (x, y) -plane is given in polar coordinates (r, θ) (where $x = r \cos \theta$, $y = r \sin \theta$) by the equation $r(\theta) = \sin \theta$. Show that

$$\frac{dy}{dx} = \tan 2\theta.$$

Another curve is given by the equation $r(\theta) = 1 + \sin^2 \theta$. Find $\frac{dy}{dx}$ at the point corresponding to $\theta = \pi/4$.

4. Find the location and nature of the stationary points of the following functions:

(a) $y = 2x^3 + 15x^2 - 84x$;

(b) $y = x^5 - 5x + 1$;

(c) $y = \log 2x + \frac{1}{x}$.

5. If c is a constant and

$$x(t) = \frac{ct}{1-t}, \quad y(t) = \frac{ct^2}{1-2t}$$

find an expression for $\frac{dy}{dx}$ in terms of x and y . Hence verify, by differentiation, that

$$\frac{1}{y} = \frac{c}{x^2} + k$$

where k is a constant, and determine k in terms of c .

6. If

$$x(s) = \cos 2s, \quad y(s) = s - \tan s,$$

show that

$$\frac{dy}{dx} = \frac{1}{2} \sqrt{\frac{1-x}{(1+x)^3}}.$$

7. Given that $xy(x+y)^a = b$, where a and b are constants, show that

$$\frac{dy}{dx} = -\frac{y}{x} \left(\frac{(a+1)x+y}{(a+1)y+x} \right).$$

8. Establish the relation

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2} \left(\frac{dy}{dx} \right)^{-3}.$$

Verify this expression for the example $y = (1+x)^{-1}$.

9. If

$$y = \left[x + 1 + \sqrt{x^2 + 2x + 2} \right]^p,$$

show that

$$\sqrt{x^2 + 2x + 2} \frac{dy}{dx} = py.$$

Differentiate this equation to show that

$$(x^2 + 2x + 2) \frac{d^2y}{dx^2} + (x + 1) \frac{dy}{dx} - p^2y = 0.$$

Now use Leibniz's formula to show that, at $x = 0$,

$$2 \frac{d^{n+2}y}{dx^{n+2}} + (2n + 1) \frac{d^{n+1}y}{dx^{n+1}} + (n^2 - p^2) \frac{d^n y}{dx^n} = 0.$$

10. Use Leibniz's formula to show that:

$$(a) \quad \frac{d^n}{dx^n} [x^3 \log x] = 6(-1)^n (n - 4)! x^{-n+3};$$
$$(b) \quad \frac{d^n}{dx^n} [(1 + x^2)\exp(x)] = 1 + n(n - 1) \quad \text{when } x = 0.$$

11. Find $f^{(n)}(x)$ if

$$f(x) = \frac{1}{x^2 + 3x + 2}.$$

12. A sheet of metal of fixed area A is to be made into a right circular cylinder with closed ends. Show that the volume of the cylinder is a maximum when its length is equal to its diameter and find this maximum volume.

13. Two passages, of widths a and b , meet at a right-angled corner. Show that the longest ladder that can be taken around the corner is of length $(a^{2/3} + b^{2/3})^{3/2}$.