M1M1: Problem Sheet 3: SOLUTIONS

Differentiation

1.

$$(a)\quad \frac{dx^3}{dx} = \lim_{\epsilon\to 0}\left(\frac{(x+\epsilon)^3 - x^3}{\epsilon}\right) = \lim_{\epsilon\to 0}\left(\frac{x^3 + 3x^2\epsilon + 3x\epsilon^2 + \epsilon^3 - x^3}{\epsilon}\right) = 3x^2.$$

$$\begin{aligned}(b)\quad \frac{dx^{1/3}}{dx} &= \lim_{\epsilon\to 0}\left(\frac{(x+\epsilon)^{1/3} - x^{1/3}}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{x^{1/3}(1+\epsilon/x)^{1/3} - x^{1/3}}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{x^{1/3}(1+\frac{\epsilon}{3x}+...) - x^{1/3}}{\epsilon}\right)\\ &= \frac{1}{3x^{2/3}}.\end{aligned}$$

$$\begin{aligned}(c)\quad \frac{d(x^2-1)^{1/2}}{dx} &= \lim_{\epsilon\to 0}\left(\frac{((x+\epsilon)^2-1)^{1/2} - (x^2-1)^{1/2}}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{(x^2-1+(2\epsilon x+\epsilon^2))^{1/2} - (x^2-1)^{1/2}}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{(x^2-1)^{1/2}\left[\left(1+\frac{2\epsilon x+\epsilon^2}{x^2-1}\right)^{1/2} - 1\right]}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{(x^2-1)^{1/2}\left[\left(1+\frac{1}{2}\left(\frac{2\epsilon x+\epsilon^2}{x^2-1}\right)+...\right) - 1\right]}{\epsilon}\right)\\ &= \frac{x}{(x^2-1)^{1/2}}.\end{aligned}$$

$$\begin{aligned}(d)\quad \frac{d\cos(x)}{dx} &= \lim_{\epsilon\to 0}\left(\frac{\cos(x+\epsilon) - \cos(x)}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{\cos(x)\cos(\epsilon) - \sin(x)\sin(\epsilon) - \cos(x)}{\epsilon}\right)\\ &= \lim_{\epsilon\to 0}\left(\frac{\cos(x)\left(1-\frac{\epsilon^2}{2!}+...\right) - \sin(x)\left(\epsilon-\frac{\epsilon^3}{3!}+...\right) - \cos(x)}{\epsilon}\right)\\ &= -\sin(x).\end{aligned}$$

$$(e)\quad \frac{d\tan(x)}{dx} = \lim_{\epsilon\to 0}\left(\frac{\tan(x+\epsilon)-\tan(x)}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\frac{\tan(x)+\tan(\epsilon)}{1-\tan(x)\tan(\epsilon)} - \tan(x)}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left[(\tan(x)+\tan(\epsilon))\left(1+\tan(x)\tan(\epsilon)+...\right)-\tan x\right]\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left[\tan(x)+\tan(\epsilon)+\tan^2(x)\tan(\epsilon)+\tan(x)\tan^2\epsilon+...-\tan(x)\right]\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\tan(\epsilon)}{\epsilon}\left(1+\tan^2(x)+\tan(x)\tan(\epsilon)+...\right)\right)$$

$$= 1+\tan^2(x) = \sec^2(x).$$

$$(f)\quad \frac{d\sin(x^{1/2})}{dx} = \lim_{\epsilon\to 0}\left(\frac{\sin((x+\epsilon)^{1/2})-\sin(x^{1/2})}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\sin(x^{1/2}(1+\epsilon/x)^{1/2})-\sin(x^{1/2})}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\sin(x^{1/2}(1+\frac{\epsilon}{2x}+...)-\sin(x^{1/2})}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\sin(x^{1/2}+\frac{\epsilon}{2x^{1/2}}+...)-\sin(x^{1/2})}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\sin(x^{1/2})\cos(\frac{\epsilon}{2x^{1/2}}+...)+\cos(x^{1/2})\sin(\frac{\epsilon}{2x^{1/2}}+...)-\sin(x^{1/2})}{\epsilon}\right)$$

$$= \lim_{\epsilon\to 0}\left(\frac{\sin(x^{1/2})(1-\frac{\epsilon^2}{8x}+...)+\cos(x^{1/2})(\frac{\epsilon}{2x^{1/2}}+...)-\sin(x^{1/2})}{\epsilon}\right)$$

$$= \frac{\cos(x^{1/2})}{2x^{1/2}}.$$

2. (a) $2x\cos(x^2)$; (b) $2\sin(x)\cos(x)$; (c) $2\cos(2x)$;
(d) $10^x = \exp(x\log(10))$ so derivative is

$$\log(10)\exp(x\log(10)) = \log(10)10^x.$$

(e) $(\sin(x))^x = \exp(x\log(\sin(x)))$ so derivative is

$$\left[\log(\sin(x)) + \frac{x\cos(x)}{\sin(x)}\right]\exp(x\log(\sin(x))) = \left[\log(\sin(x)) + \frac{x\cos(x)}{\sin(x)}\right](\sin(x))^x.$$

(f) $\cos(x)\sec^2(\sin(x))$.

(g) $y = \cot^{-1}(x)$ so $\cot(y) = x$. Therefore, taking a derivative of this last equation with respect to x:

$$-\frac{dy}{dx}\mathrm{cosec}^2(y) = 1$$

which implies

$$\frac{d\cot^{-1}(x)}{dx} = \frac{dy}{dx} = -\frac{1}{\mathrm{cosec}^2(y)} = -\frac{1}{1+\cot^2(y)} = -\frac{1}{1+x^2}.$$

(h) $(6x+5)\exp(3x^2+5x+2)$.

(i) $-\exp(-x)\cosh(2x) + 2\exp(-x)\sinh(2x)$.

(j) $\frac{\exp(x)}{x} + \exp(x)\log(x)$.

(k) $y = \sin^{-1}(x)$ so $\sin(y) = x$. Therefore, taking a derivative of this last equation with respect to x:

$$\frac{dy}{dx}\cos(y) = 1$$

which implies

$$\frac{d\sin^{-1}(x)}{dx} = \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}.$$

(l) $\frac{\sin(x)}{\cos^2(x)}$.

(m) Function can be written $\log|1+\sin(x)| - \log|\cos(x)|$ so derivative is

$$\frac{\cos(x)}{1+\sin(x)} + \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)}.$$

(n) $\frac{x}{(x^2-1)^{1/2}}$.

(o) $-\frac{1}{(x^2-1)^{3/2}}$.

(p) $-\frac{3x}{(x^2-1)^{5/2}}$.

(q) $\frac{\sin^{-1}(x)}{x} + \frac{\log(x)}{(1-x^2)^{1/2}}$.

(r) $\frac{1}{1+x^2}$ (similar to (g)).

3. We know $x(\theta) = r(\theta)\cos(\theta)$ and $y(\theta) = r(\theta)\sin(\theta)$ where $r(\theta) = \sin(\theta)$. Now, $x(\theta) = \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$ and $y(\theta) = \sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$ so

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta).$$

Similarly, if $r(\theta) = 1 + \sin^2(\theta)$ then

$$x(\theta) = (1+\sin^2(\theta))\cos(\theta) \quad \text{and} \quad y(\theta) = (1+\sin^2(\theta))\sin(\theta)$$

so that

$$\begin{aligned}\frac{dx}{d\theta} &= 2\sin(\theta)\cos^2(\theta) - (1+\sin^2(\theta))\sin(\theta),\\ \frac{dy}{d\theta} &= 2\sin^2(\theta)\cos(\theta) + (1+\sin^2(\theta))\cos(\theta).\end{aligned}$$

On use of the facts that $\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$ we get

$$\left.\frac{dy}{dx}\right|_{\pi/4} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\pi/4} = -5.$$

4.(a) On differentiation,

$$\frac{dy}{dx} = 6x^2 + 30x - 84 = 6(x+7)(x-2),$$

so there are stationary points at $x = 2, -7$. Now

$$\frac{d^2y}{dx^2} = 12x + 30,$$

so at $x = 2$, $d^2y/dx^2 > 0$ so we have a minimum, while at $x = -7$, $d^2y/dx^2 < 0$ so we have a maximum.

(b) On differentiation

$$\frac{dy}{dx} = 5x^4 - 5 = 5(x^4 - 1),$$

so there are stationary points at $x = \pm 1$. Now

$$\frac{d^2y}{dx^2} = 20x^3,$$

so at $x = 1$, $d^2y/dx^2 > 0$ so we have a minimum, while at $x = -1$, $d^2y/dx^2 < 0$ so we have a maximum.

On differentiation,

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2},$$

so there is a stationary point at $x = 1$. Now

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} + \frac{2}{x^3},$$

so at $x = 1$, $d^2y/dx^2 > 0$ so we have a minimum.

5. First note that

$$\begin{aligned}
\frac{dx}{dt} &= \frac{c}{1-t} + \frac{ct}{(1-t)^2} = \frac{c}{(1-t)^2}, \\
\frac{dy}{dt} &= \frac{2ct}{1-2t} + \frac{2ct^2}{(1-2t)^2} = \frac{2ct(1-t)}{(1-2t)^2}.
\end{aligned}$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t(1-t)^3}{(1-2t)^2} = \frac{2cy^2}{x^3}.$$

If

$$\frac{1}{y} = \frac{c}{x^2} + k$$

then, by differentiation with respect to x,

$$-\frac{1}{y^2}\frac{dy}{dx} = -\frac{2c}{x^3},$$

or

$$\frac{dy}{dx} = \frac{2cy^2}{x^3},$$

which is consistent with what has already been deduced. But

$$\begin{aligned}
\frac{1}{y} &= \frac{1-2t}{ct^2} = \frac{1}{c}\left(\frac{1}{t^2} - \frac{2}{t}\right), \\
\frac{c}{x^2} &= \frac{c(1-t)^2}{c^2t^2} = \frac{1}{c}\left(\frac{1}{t^2} - \frac{2}{t} + 1\right).
\end{aligned}$$

By inspection, we see that $k = -1/c$.

6. On differentiation and use of trigonometric identities,

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{1 - \sec^2(s)}{-2\sin(2s)} = \frac{\sin(s)}{4\cos^3(s)}.$$

But,

$$\sin(s) = \left(\frac{1-\cos(2s)}{2}\right)^{1/2} = \left(\frac{1-x}{2}\right)^{1/2},$$

$$\cos(s) = \left(\frac{1+\cos(2s)}{2}\right)^{1/2} = \left(\frac{1+x}{2}\right)^{1/2}.$$

On use of these identities, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{(1+x)^3}\right)^{1/2}.$$

7. On differentiation of $xy(x+y)^a = b$ with respect to x,

$$y(x+y)^a + x\frac{dy}{dx}(x+y)^a + axy(x+y)^{(a-1)}\left(1 + \frac{dy}{dx}\right) = 0.$$

On rearrangement,

$$\frac{dy}{dx} = -\frac{y}{x}\left(\frac{(a+1)x+y}{(a+1)y+x}\right).$$

8. From lectures it is known that

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}.$$

On differentiation of this expression with respect to y and use of the chain rule:

$$\begin{aligned}\frac{d^2x}{dy^2} &= \frac{dx}{dy}\frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1} \\ &= -\frac{dx}{dy}\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2} \\ &= -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}.\end{aligned}$$

Now

$$y = \frac{1}{1+x}, \quad \frac{dy}{dx} = -\frac{1}{(1+x)^2} = -y^2, \quad \frac{d^2y}{dx^2} = \frac{2}{(1+x)^3} = 2y^3.$$

But,

$$y(1+x) = 1, \quad \text{so} \quad x = \frac{1-y}{y} = \frac{1}{y} - 1.$$

Differentiating the previous equation twice with respect to y:

$$\frac{d^2x}{dy^2} = \frac{2}{y^3}.$$

But, also

$$-\left(\frac{dy}{dx}\right)^{-3} \frac{d^2y}{dx^2} = \frac{1}{y^6} 2y^3 = \frac{2}{y^3},$$

which is consistent and provides a check of the formula.

9. On differentiation with respect to x:

$$\frac{dy}{dx} = \left(1 + \frac{x+1}{\sqrt{x^2+2x+2}}\right) \frac{py}{x+1+\sqrt{x^2+2x+2}}.$$

On rearrangement,

$$\sqrt{x^2+2x+2}\frac{dy}{dx} = py.$$

Differentiating the previous equation with respect to x,

$$\frac{x+1}{\sqrt{x^2+2x+2}}\frac{dy}{dx} + \sqrt{x^2+2x+2}\frac{d^2y}{dx^2} = p\frac{dy}{dx} = p^2\frac{y}{\sqrt{x^2+2x+2}}.$$

On rearrangement,

$$(x^2+2x+2)\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} - p^2y = 0.$$

Now differentiate the previous equation n-times with respect to x. By the Leibniz rule,

$$\frac{d^n}{dx^n}\left((x^2+2x+2)\frac{d^2y}{dx^2}\right) = (x^2+2x+2)\frac{d^{(n+2)}y}{dx^{(n+2)}} + n(2x+2)\frac{d^{(n+1)}y}{dx^{(n+1)}} + \frac{2n(n-1)}{2}\frac{d^ny}{dx^n}$$

while

$$\frac{d^n}{dx^n}\left((x+1)\frac{dy}{dx}\right) = (x+1)\frac{d^{(n+1)}y}{dx^{(n+1)}} + n\frac{d^n y}{dx^n}.$$

On use of these expressions, evaluated at $x = 0$, gives the result.

10. (a) By the Leibniz rule,

$$\frac{d^n}{dx^n}\left[x^3\log(x)\right] = x^3\frac{d^n\log(x)}{dx^n} + n3x^2\frac{d^{(n-1)}\log(x)}{dx^{(n-1)}} + \frac{n(n-1)}{2!}6x\frac{d^{(n-2)}\log(x)}{dx^{(n-2)}}$$
$$+ \frac{n(n-1)(n-2)}{3!}6\frac{d^{(n-3)}\log(x)}{dx^{(n-3)}}.$$

But

$$\frac{d\log(x)}{dx} = \frac{1}{x}; \quad \frac{d^2\log(x)}{dx^2} = -\frac{1}{x^2}; \quad \frac{d^3\log(x)}{dx^3} = \frac{2}{x^3}; \quad \text{etc....}$$

therefore, pattern clearly gives

$$\frac{d^j\log(x)}{dx^j} = (-1)^{(j+1)}\frac{(j-1)!}{x^j}, \quad \text{for } j \geq 1.$$

On use of this,

$$\frac{d^n}{dx^n}\left[x^3\log(x)\right] = x^3\frac{(n-1)!}{x^n}(-1)^{(n+1)} + 3x^2 n\frac{(n-2)!}{x^{(n-1)}}(-1)^n$$
$$+ 3xn(n-1)\frac{(n-3)!}{x^{(n-2)}}(-1)^{(n-1)} + n(n-1)(n-2)\frac{(n-4)!}{x^{(n-3)}}(-1)^{(n-2)}.$$

Rearrangement gives

$$\frac{d^n}{dx^n}\left[x^3\log(x)\right] = 6\frac{(n-4)!(-1)^n}{x^{(n-3)}}.$$

(b) By the Leibniz rule,

$$\frac{d^n}{dx^n}\left[(1+x^2)\exp(x)\right] = (1+x^2)\exp(x) + 2nx\exp(x) + 2\frac{n(n-1)}{2}\exp(x)$$
$$= \exp(x)\left(1 + x^2 + 2nx + n(n-1)\right).$$

Therefore, at $x = 0$,

$$\frac{d^n}{dx^n}\left[(1+x^2)\exp(x)\right] = 1 + n(n-1).$$

11. Can either use the Leibniz rule **or** the partial fraction decomposition gives

$$f(x) = \frac{1}{x+1} - \frac{1}{x+2}.$$

Now,

$$\begin{aligned} f'(x) &= -\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2}; \\ f''(x) &= \frac{2}{(x+1)^3} - \frac{2}{(x+2)^3}; \\ f'''(x) &= -\frac{3*2}{(x+1)^4} + \frac{3*2}{(x+2)^4}; \quad \text{etc.} \end{aligned}$$

so pattern is clearly

$$\frac{d^n f(x)}{dx^n} = (-1)^n n! \left(\frac{1}{(x+1)^{(n+1)}} - \frac{1}{(x+2)^{(n+1)}} \right).$$

12. Let r be the radius of the cylinder and let h be its length. Then the volume $V = \pi r^2 h$ and the surface area $A = 2\pi r^2 + 2\pi r h$. The volume will be a maximum when

$$\frac{dV}{dr} = 2\pi r h + \pi r^2 \frac{dh}{dr} = 0,$$

so that

$$\frac{dh}{dr} = -\frac{2h}{r}.$$

But since A is fixed, we also have

$$\frac{dA}{dr} = 2\pi \left(2r + h + r\frac{dh}{dr} \right) = 0.$$

But using the fact that $\frac{dh}{dr} = -\frac{2h}{r}$ in the previous equation gives

$$h = 2r,$$

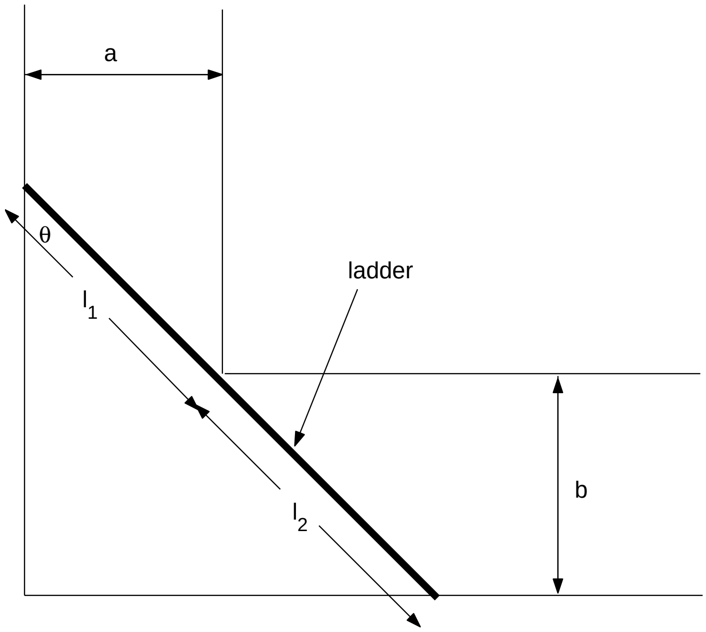

Figure 1: Diagram for question 13

so, at the maximum volume, the length is equal to the diameter. The maximum volume $V_{max} = \pi r^2(2r) = 2\pi r^3$. But

$$A = 2\pi r^2 + 4\pi r^2 = 6\pi r^2,$$

so that

$$V_{max} = 2\pi \left(\frac{A}{6\pi}\right)^{3/2}.$$

13. The length L of the ladder as a function of the angle θ (defined in the figure) is

$$L = L(\theta) = l_1(\theta) + l_2(\theta) = \frac{a}{\sin(\theta)} + \frac{b}{\cos(\theta)} = a\mathrm{cosec}(\theta) + b\sec(\theta).$$

Differentiating $L(\theta)$ with respect to θ:

$$\frac{dL}{d\theta} = -\frac{a\cos(\theta)}{\sin^2(\theta)} + \frac{b\sin(\theta)}{\cos^2(\theta)}.$$

This vanishes (so that length is a maximum) when

$$\tan(\theta) = \left(\frac{a}{b}\right)^{1/3}.$$

But

$$\sec^2(\theta) = 1 + \tan^2(\theta) = \frac{b^{2/3} + a^{2/3}}{b^{2/3}},$$
$$\mathrm{cosec}^2(\theta) = 1 + \cot^2(\theta) = \frac{b^{2/3} + a^{2/3}}{a^{2/3}}.$$

Putting these into the expression for the length $L(\theta)$ gives the maximum length as

$$\left(a^{2/3} + b^{2/3}\right)^{3/2}.$$