M1M1: Problem Sheet 4: SOLUTIONS

Curve sketching

1. First, rewrite this rational function in a more convenient form:

$$y = \frac{x+3}{2x+1} = \frac{x+3}{2(x+1/2)} = \frac{x+1/2+5/2}{2(x+1/2)}$$
$$= \frac{1}{2} + \frac{5}{4(x+1/2)}.$$

$x = -1/2$ is a vertical asymptote.

As $x \to \infty$, $y \to 1/2$ which is therefore a horizontal asymptote.

See Figure 1.

2.(a) $x^2 - 6x + 8 = 0$ is equivalent to $(x-2)(x-4) = 0$. Graph is a parabola meeting x-axis at $x = 2, 4$. See Figure 2.

(b) First note that as $x \to \pm\infty$, $y \to 0$. Note also that function is odd, i.e., $y(-x) = -y(x)$. Also,

$$\frac{dy}{dx} = (1 - 2x^2)\exp(-x^2).$$

There are therefore stationary points at $x = \pm\frac{1}{\sqrt{2}}$. See Figure 3.

(c) Note that

$$\frac{1}{\sqrt{2}}\cos(x) + \frac{1}{\sqrt{2}}\sin(x) = \cos(x - \pi/4).$$

Graph is therefore a usual cosine graph displaced to the right by $\pi/4$. See Figure 4.

3. Rewrite the rational function as

$$y = \frac{x^2+2x+5}{x+1} = \frac{x(x+1) - x + 2x + 5}{x+1} = x + \frac{x+5}{x+1} = x + \frac{x+1+4}{x+1}$$
$$= x + 1 + \frac{4}{x+1}.$$

Note that as $x \to \pm\infty$, $y \to x + 1$ which is an asymptote.

There is a vertical asymptote at $x = -1$.

Also,

$$\frac{dy}{dx} = 1 - \frac{4}{(x+1)^2},$$

so there are stationary points at $x = -3, 1$. Note also

$$\frac{d^2y}{dx^2} = \frac{8}{(x+1)^3},$$

so this never vanishes (there are therefore no points of inflexion). See Figure 5.

4. First, rewrite in partial fraction form:

$$y = \frac{x^2 - 2x}{x - 3} = \frac{x(x-3) + 3x - 2x}{x - 3} = x + \frac{x}{x-3} = x + 1 + \frac{3}{x-3}.$$

As $x \to \pm\infty$, $y \to x + 1$, which is an asymptote.

There is a vertical asymptote at $x = 3$.

Note that

$$\frac{dy}{dx} = 1 - \frac{3}{(x-3)^2},$$

so there are stationary points at $x = 3 \pm \sqrt{3}$. See Figure 5.

5. First note that curve is invariant if $y \mapsto -y$ which means it is reflectionally symmetric with respect to the x-axis.

Note also that the curve is only defined provided $x \geq 1$ and $-1 \leq x \leq 0$.

For small x, $y^2 \sim -x$ so graph is locally parabolic near origin.

For large x, $y^2 \sim x^3$ so that $y \sim \pm x^{3/2}$.

Note also that

$$2y\frac{dy}{dx} = 3x^2 - 1,$$

so that

$$\frac{dy}{dx} = \frac{3x^2 - 1}{\sqrt{x(x^2-1)}}.$$

So stationary points are at $x = \pm\frac{1}{\sqrt{3}}$ while $\frac{dy}{dx} \to \infty$ at $x = 0, \pm 1$. There must be points at inflexion for some $x > 0$. To check this, note that

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 6x,$$

so that

$$4y\frac{d^2y}{dx^2} = \frac{3x^4 - 6x^2 - 1}{x(x^2-1)},$$

which vanishes at $x^2 = 1 + \frac{2}{\sqrt{3}}$. See Figure 6.

6. Using polar coordinates, $x = r\cos\theta,\ y = r\sin\theta$, so that

$$\begin{aligned} 3r + r\cos\theta &= 2, \\ 3r + x &= 2, \\ 3r &= 2 - x, \\ 9r^2 &= (2-x)^2 = 4 - 4x + x^2. \end{aligned}$$

On use of $r^2 = x^2 + y^2$, rearrangement yields

$$\frac{(x+1/4)^2}{9/16} + \frac{y^2}{1/2} = 1.$$

This is an ellipse, centred at $(-1/4, 0)$ with semi-major axis $3/4$ and semi-minor axis $1/\sqrt{2}$. See Figure 6.

7. Note first that parametrization is 2π-periodic so it is enough to consider $0 \le t \le 2\pi$.

Note also that curve always lies within $-a \le x \le a$ and $-b \le y \le b$.

Also,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3b\sin^2(t)\cos(t)}{-3a\cos^2(t)\sin(t)} = -\frac{b}{a}\tan(t),$$

so gradient is zero at $t = 0, \pi, 2\pi$ and infinite at $t = \pi/2, 3\pi/2$.

Note also that graph is in first quadrant for $0 \le t \le \pi/2$, second quadrant for $\pi/2 \le t \le \pi$, the third quadrant for $\pi \le t \le 3\pi/2$ and the fourth quadrant for $3\pi/2 \le t \le 2\pi$. See Figure 7.

8. (a) Note that

$$f(x) = \frac{x^3-1}{x^3+1} = \frac{x^3+1-2}{x^3+1} = 1 - \frac{2}{x^3+1}.$$

But $x^3 + 1 = (x+1)(x^2 - x + 1)$, so partial fraction form is

$$f(x) = 1 + \frac{A}{x+1} + \frac{Bx + C}{(x^2 - x + 1)}.$$

Some algebra gives

$$f(x) = 1 - \frac{2}{3(x+1)} + \frac{2}{3}\frac{(x-2)}{(x^2-x+1)}.$$

(b) Now,

$$f'(x) = \frac{6x^2}{(x^3+1)^2}.$$

Therefore there is a stationary point at $x = 0$. To classify it, consider

$$f''(x) = \frac{12x(1-2x^3)}{(x+3+1)^3},$$

therefore $f''(0) = 0$ so need to look at $f'''(0)$. Some algebra gives

$$f'''(x) = \frac{(x^3+1)^3(12-96x^3) - 9x^2(12x-24x^4)(x^3+1)^2}{(x^3+1)^6},$$

so $f'''(0) \neq 0$. Therefore $x = 0$ is a point of inflexion (as well as being a stationary point).

(c) Note from the above that $f''(x) = 0$ when $x = \frac{1}{2^{1/3}}$ so this is also a possible point of inflexion. Also from above, $f'''(1/2^{1/3}) \neq 0$, so this is another point of inflexion. See Figure 8.

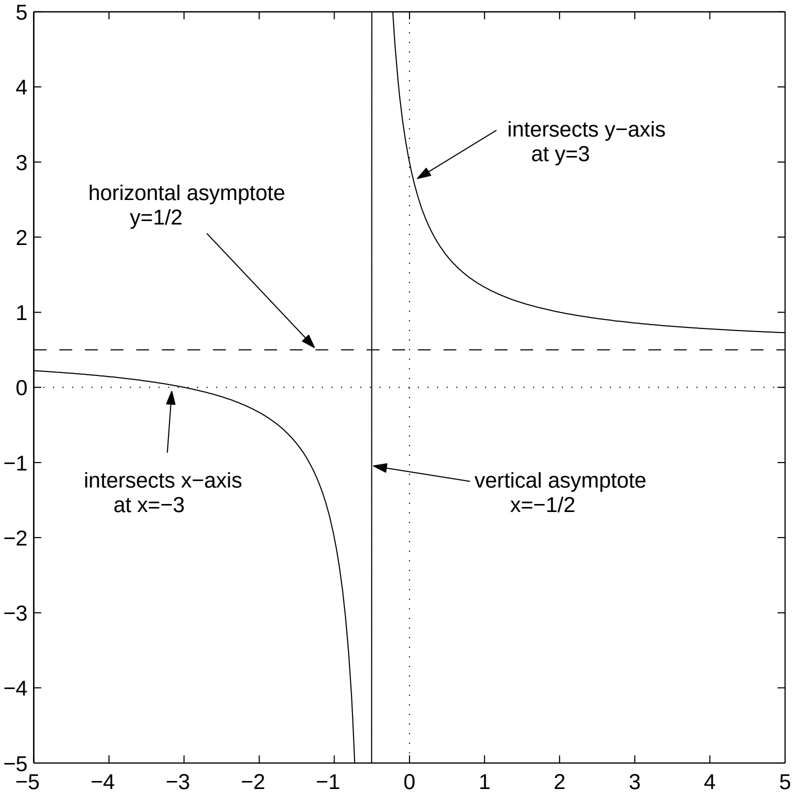

Figure 1: Graph for Q1

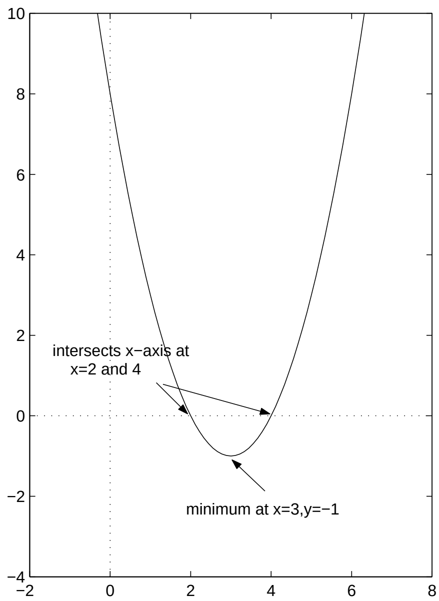

Figure 2: Graph for Q2(a)

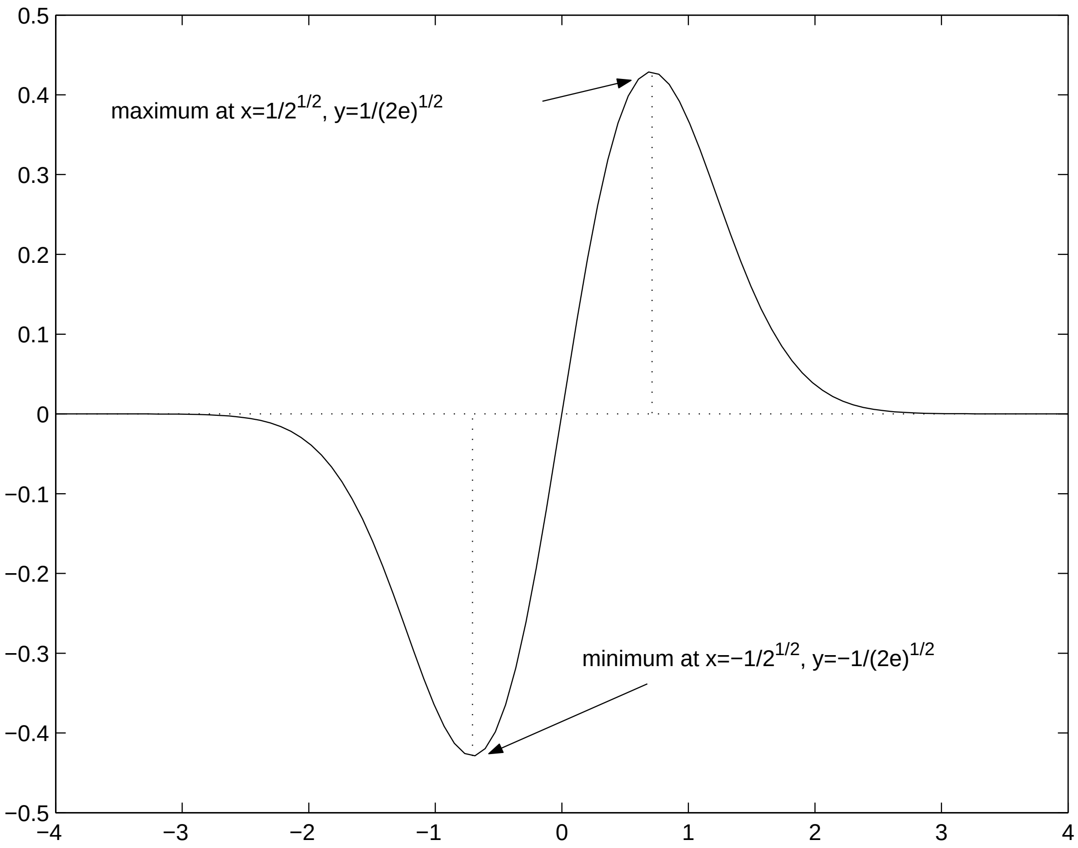

Figure 3: Graph for Q2(b)

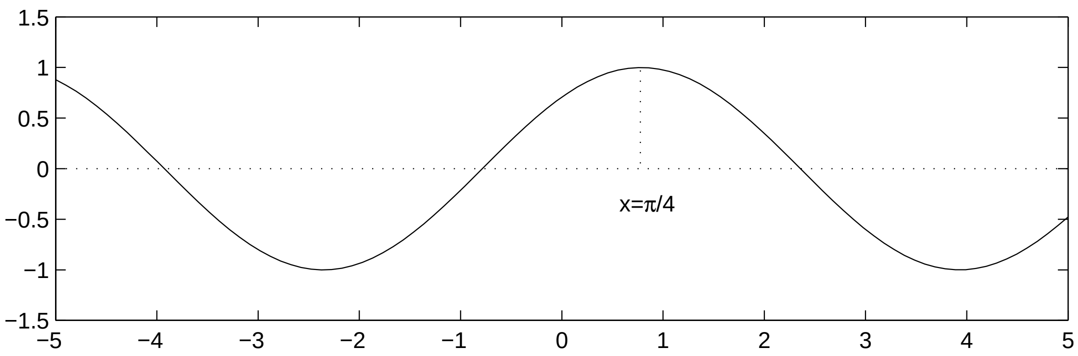

Figure 4: Graph for Q2(c)

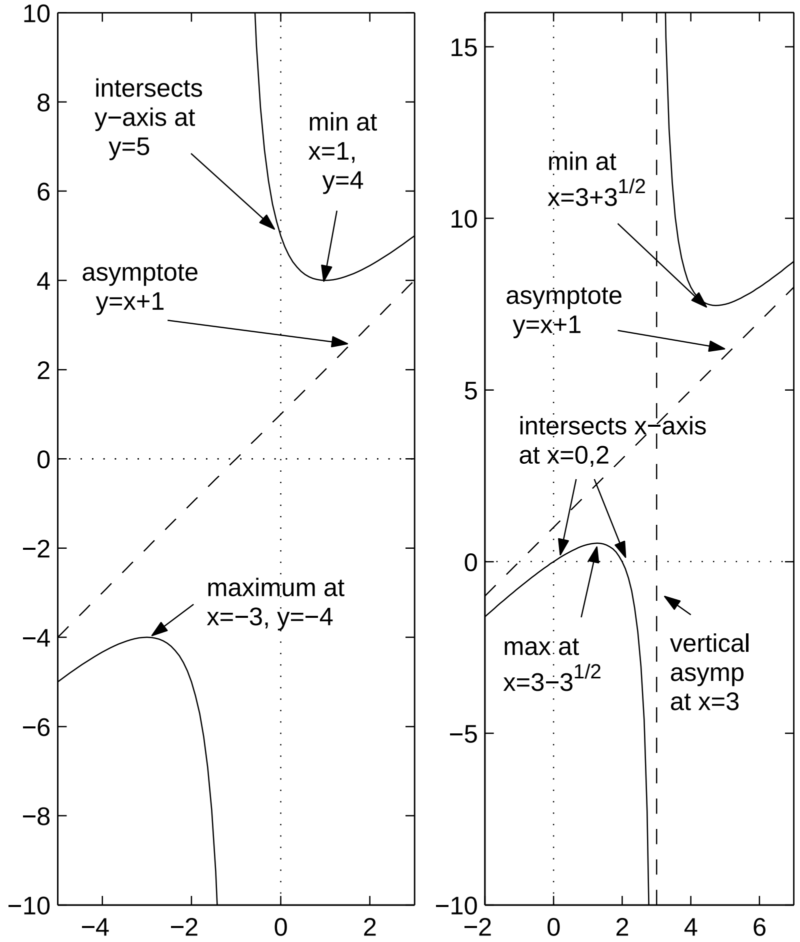

Figure 5: Graphs for Q3 (left) and Q4 (right)

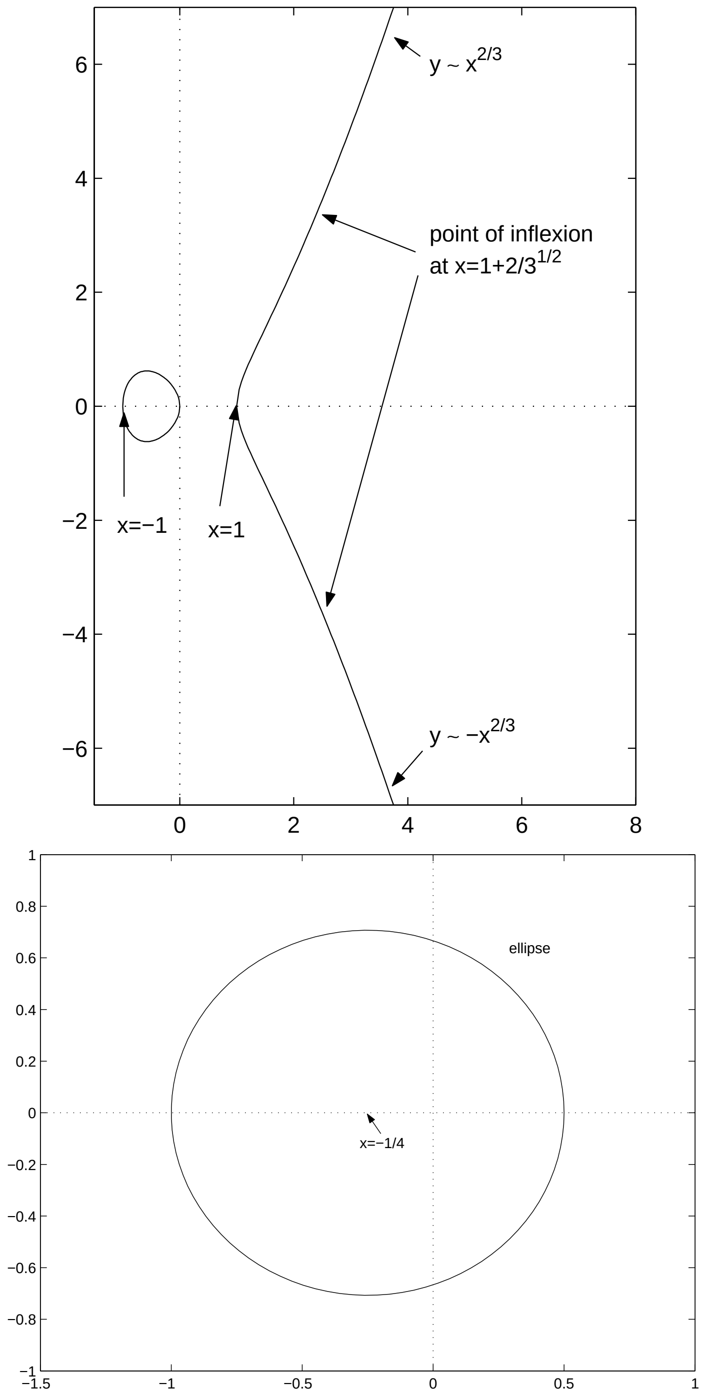

Figure 6: Graphs for Q5 (top) and Q6 (bottom)

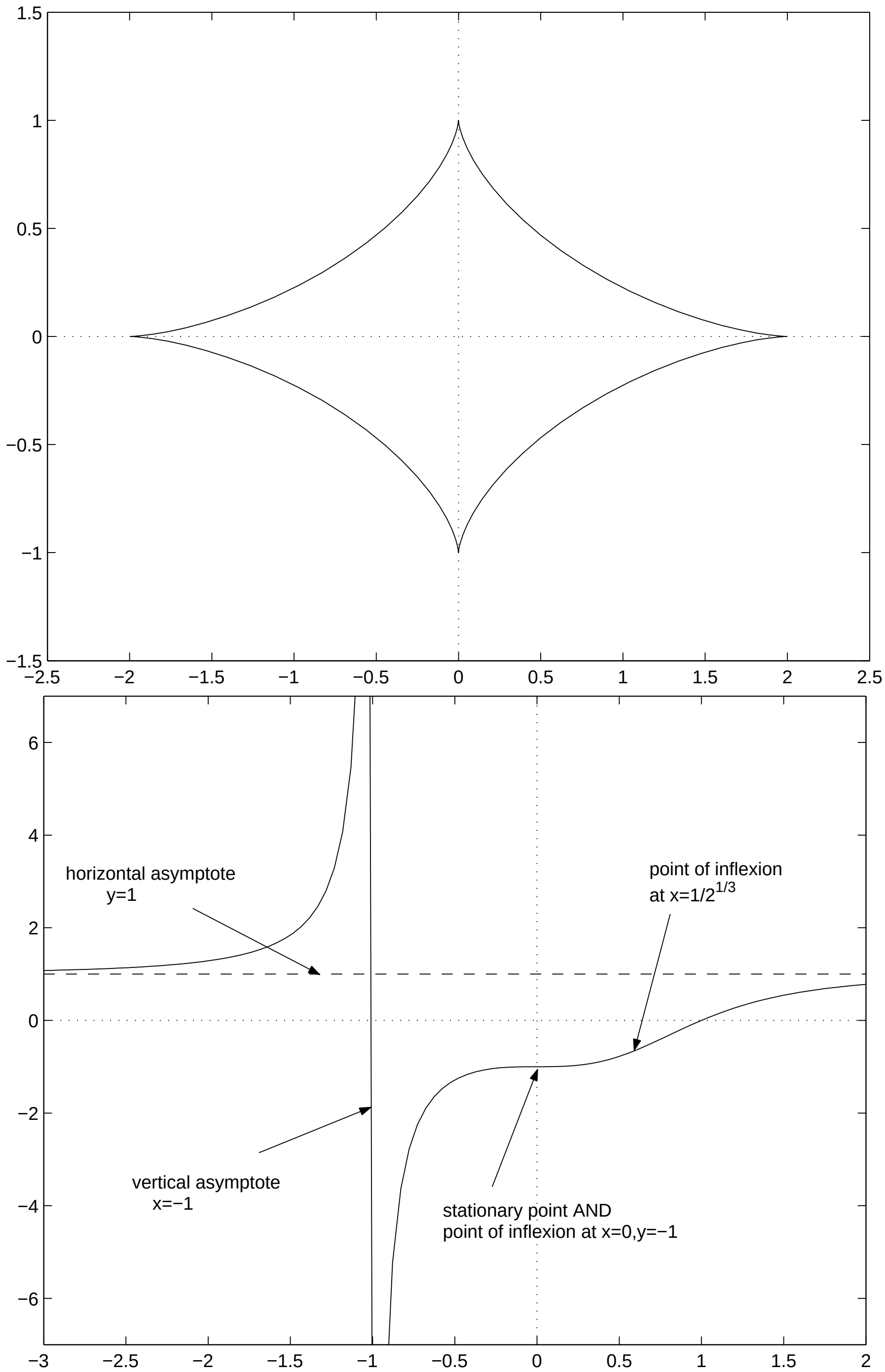

Figure 7: Graphs for Q7 (top) and Q8 (bottom)