M1M1: Problem Sheet 5

Mean value theorem and Taylor series

1. Consider the function $f(x) = x^3 - 8x - 5$ in the domain $1 \leq x \leq 4$. Find a number c, with $1 < c < 4$, such that

$$\frac{f(4) - f(1)}{3} = f'(c).$$

2. Let $f(x) = \frac{4}{x}$ in the domain $-1 \leq x \leq 4$. Show that there is no number c such that

$$\frac{f(4) - f(-1)}{5} = f'(c).$$

Why does this not contradict the mean value theorem?

3. Use the mean value theorem to show that

$$\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}.$$

4. Apply the mean value theorem to $\tan^{-1}(x)$ to show that

$$\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$$

where $a < b$. Hence obtain the value of $\tan^{-1}(21/20)$ to 2 decimal places of accuracy.

5. Use the first few terms of an appropriate binomial expansion to estimate the value of $8.1^{1/3}$, giving your answer correct to four decimal places.

6. Find the first few terms in the Taylor series of $\sin(x + a)$ about $x = 0$ and hence verify the identity

$$\sin(x + a) = \sin x \cos a + \sin a \cos x$$

7. Show that

$$(a)\ \ \tan(x + \pi/4) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + ...;$$

$$(b)\ \ \log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + ...$$

8. Find the first three non-zero terms of the Taylor expansions about $x = 0$ of the functions

$$(a)\ \exp(x)\cos x; \quad (b)\ \tan^{-1}(x); \quad (c)\ \sec x.$$

9. Given that $y(x) = \sin^{-1}(x)$, show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

By differentiating this equation n times, show that

$$\frac{d^{n+2}y}{dx^{n+2}} - n^2\frac{d^ny}{dx^n} = 0$$

at $x = 0$. Use this relation to derive the Taylor expansion

$$\sin^{-1}(x) = x + \frac{x^3}{3!} + \frac{3^2x^5}{5!} + ...$$

10. Let

$$y(x) = \sin(m\sin^{-1}(x))$$

where m is some real number. By differentiating this equation n times, show that

$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2 - n^2)\frac{d^ny}{dx^n} = 0.$$

Now set $x = 0$ and hence derive the following Taylor expansion about $x = 0$ for $y(x)$:

$$y(x) = mx + m(1 - m^2)\frac{x^3}{3!} + m(1 - m^2)(9 - m^2)\frac{x^5}{5!} + ...$$

Show that this series converges for $|x| < 1$.

11. Show that $y(x) = \tan(x)$ satisfies the equation

$$\frac{dy}{dx} = 1 + y^2.$$

By repeated differentiation of this equation, find the higher derivatives of $y(x)$ and hence determine the first three non-zero terms of the Taylor expansion of $\tan x$ about $x = 0$. Check your answer by using the definition

$$\tan x = \frac{\sin x}{\cos x}$$

and using the well-known expansions of $\sin x$ and $\cos x$.

12. Derive the Taylor series about $x = 0$ for the function

$$\log\left(\frac{1+x}{1-x}\right).$$

State its radius of convergence and use the series to obtain the value of $\log(5/3)$ to 4 decimal places of accuracy.

13. Use a Taylor series expansion to show that, when h is small,

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h}$$

with an error of order

$$\frac{h^2 f'''(a)}{6}.$$

Show also that

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

with an error of order

$$\frac{h^2 f''''(a)}{12}.$$

Now let $f(x) = \sin x$ and $h = \pi/12$. From the above equations, find the values of $f'(\pi/4)$ and $f''(\pi/4)$ and compare with the exact values.