M1M1: Problem Sheet 5: SOLUTIONS

Mean value theorem and Taylor series

1. Note that $f'(x) = 3x^2 - 8$ and that $f(1) = -12, f(4) = 27$. Therefore

$$\frac{f(4) - f(1)}{3} = 13.$$

But $13 = f'(c) = 3c^2 - 8$ for some c, which implies $c^2 = 7$. Thus, $c = \sqrt{7}$.

2. Note that $f'(x) = -\frac{4}{x^2}$ while $f(-1) = -4, f(4) = 1$. But

$$\frac{f(4) - f(-1)}{5} = 1,$$

but this cannot equal $f'(c)$ for any c because $f'(c) < 0$ for all c.

This does not contradict the mean value theorem because, for theorem to apply, function must be continuous and differentiable throughout the interval. This $f(x)$ is singular at $x = 0$.

3. Let $f(x) = \sqrt{x}$ so that $f'(x) = \frac{1}{2\sqrt{x}}$. Applying the mean value theorem, there exists some c with $64 < c < 66$ such that

$$\frac{f(66) - f(64)}{2} = f'(c).$$

But, for this range of c,

$$\frac{1}{2\sqrt{66}} \leq f'(c) \leq \frac{1}{2\sqrt{64}},$$

so that

$$\frac{1}{2\sqrt{66}} \leq \frac{f(66) - f(64)}{2} \leq \frac{1}{2\sqrt{64}}.$$

On rearrangement, we get

$$\frac{1}{\sqrt{66}} \leq f(66) - f(64) \leq \frac{1}{\sqrt{64}}.$$

But, $\frac{1}{9} < \frac{1}{\sqrt{66}}$ so

$$\frac{1}{9} \leq \sqrt{66} - 8 \leq \frac{1}{8}.$$

4. Let $f(x) = \tan^{-1}(x)$. The mean value theorem implies that there exists c with $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

But, $f'(x) = \frac{1}{1+x^2}$, so

$$\frac{1}{1+b^2} < f'(c) < \frac{1}{1+a^2}.$$

Therefore,

$$\frac{1}{1+b^2} < \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} < \frac{1}{1+a^2}$$

which, after multiplication by the positive quantity $b - a$, gives the result. Let $a = 1$ and $b = 21/20$. Then $b - a = 0.05$. Applying the above result

$$\tan^{-1}(1) + \frac{0.05}{1+1.05^2} \leq \tan^{-1}\left(\frac{21}{20}\right) \leq \tan^{-1}(1) + \frac{0.05}{1+1}.$$

Therefore

$$\tan^{-1}\left(\frac{21}{20}\right) \approx \frac{\pi}{4} + 0.025.$$

5. Let

$$f(x) = \left(8+x\right)^{1/3} = \left(8\left(1+\frac{x}{8}\right)\right)^{1/3} = 2\left(1+\frac{x}{8}\right)^{1/3}.$$

On use of the binomial series expansion,

$$f(8+0.1) = 2\left(1 + \frac{0.1}{24} - \frac{0.1^2}{9*64} + \frac{10*0.1^3}{27*6*8^3} + ...\right) = 2.0083.$$

6. Let

$$f(x) = \sin(x+a) = f(0) + xf'(0) + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ...$$

But

$$f(0) = \sin a;\ f'(0) = \cos a;\ f''(0) = -\sin a;\ f'''(0) = -\cos a; ...$$

Therefore, it is clear that

$$\sin(x+a) = \sin a\left(1 - \frac{x^2}{2!} + ...\right) + \cos a\left(x - \frac{x^3}{3!} + ...\right) = \sin a \cos x + \cos a \sin x.$$

7.(a) By repeated differentiation,

$$f(x) = \tan\left(x + \frac{\pi}{4}\right); \quad f'(x) = \sec^2\left(x + \frac{\pi}{4}\right);$$
$$f''(x) = 2\sin\left(x + \frac{\pi}{4}\right)\sec^3\left(x + \frac{\pi}{4}\right);$$
$$f'''(x) = 2\sec^2\left(x + \frac{\pi}{4}\right) + 6\sin^2\left(x + \frac{\pi}{4}\right)\sec^4\left(x + \frac{\pi}{4}\right),$$

so that $f(0) = 1$; $f'(0) = 2$; $f''(0) = 4$; $f'''(0) = 16$. Therefore on use of the general Taylor series formula

$$\tan\left(x + \frac{\pi}{4}\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + ...$$

(b) By repeated differentiation,

$$f(x) = \log(1 + \sin x); \; f'(x) = \frac{\cos x}{1 + \sin x};$$
$$f''(x) = -\frac{\sin x}{(1 + \sin x)} - \frac{\cos^2 x}{(1 + \sin x)^2};$$
$$f'''(x) = -\frac{\cos x}{1 + \sin x} + \frac{3\cos x \sin x}{(1 + \sin x)^2} + \frac{2\cos^3 x}{(1 + \sin x)^3}.$$

Therefore $f(0) = 0$; $f'(0) = 1$; $f''(0) = -1$; $f'''(0) = 1$ so that, on use of the Taylor series formula,

$$\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + ...$$

8. (a) By repeated differentiation

$$f(x) = e^x \cos x; \; f'(x) = -e^x \sin x + e^x \cos x;$$
$$f''(x) = -2e^x \sin x; \quad f'''(x) = -2e^x \sin x - 2e^x \cos x.$$

Therefore, $f(0) = 1$; $f'(0) = 1$; $f''(0) = 0$; $f'''(0) = -2$ so that

$$e^x \cos x = 1 + x - \frac{x^3}{3} + ...$$

(b) By repeated differentiation,

$$f(x) = \tan^{-1} x;\ f'(x) = \frac{1}{1+x^2};$$
$$f''(x) = -\frac{2x}{(1+x^2)^2};\ f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3};$$
$$f^{iv}(x) = \frac{24x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4};$$
$$f^{v}(x) = \frac{24}{(1+x^2)^3} - \frac{6*48*x^2}{(1+x^2)^4} + \frac{8*48x^4}{(1+x^2)^5}$$

Therefore $f(0) = 0;\ f'(0) = 1;\ f''(0) = 0;\ f'''(0) = -2;\ f^{iv}(0) = 0;\ f^{v}(0) = 24$ so

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + ...$$

(c) By repeated differentiation,

$$f(x) = \sec x;\quad f'(x) = \sec x \tan x;$$
$$f''(x) = \sec x + 2\sec x \tan^2 x;$$
$$f'''(x) = \sec x \tan x + 2\sec x \tan^3 x + 4\sec^3 x \tan x;$$
$$f^{iv}(x) = \sec x \tan^2 x + \sec^3 x + 2\sec x \tan^4 x + 6\sec^3 x \tan^2 x$$
$$+ 12\sec^3 x \tan^2 x + 4\sec^5 x.$$

Therefore $f(0) = 1;\ f'(0) = 0;\ f''(0) = 1;\ f'''(0) = 0;\ f^{iv}(0) = 5$ so that

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + ...$$

9. Let

$$y(x) = \sin^{-1} x;\ y'(x) = \frac{1}{\sqrt{1-x^2}};$$
$$y''(x) = \frac{x}{(1-x^2)^{3/2}}$$
$$= \frac{x}{(1-x^2)}\frac{dy}{dx}.$$

Therefore

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

By the Leibniz rule, the n-th derivative of this equation is

$$(1-x^2)y^{n+2}(x) - 2xny^{n+1}(x) - n(n-1)y^n(x) - xy^{n+1}(x) - ny^n(x) = 0.$$

Thus, at $x = 0$, we have

$$y^{n+2}(0) = n^2 y^n(0)$$

Note also that $y(0) = 0$, $y'(0) = 1$. It follows that all even coefficients are zero, while $y'''(0) = y'(0) = 1$ and $y^v(0) = 3^2 y'''(0) = 3^2$ so that

$$\sin^{-1}(x) = x + \frac{x^3}{3!} + \frac{3^2 x^5}{5!} + ...$$

10. Let

$$y(x) = \sin(m\sin^{-1}(x));\; y'(x) = \frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}(x))$$

$$y''(x) = \frac{mx}{(1-x^2)^{3/2}}\cos(m\sin^{-1}(x)) - \frac{m^2}{(1-x^2)}\sin(m\sin^{-1}(x))$$

Therefore

$$y'' = \frac{x}{1-x^2}y' - \frac{m^2}{1-x^2}y$$

or

$$(1-x^2)y'' - xy' + m^2 y = 0.$$

Now take the n-th derivative,

$$\frac{d^n}{dx^n}\left((1-x^2)y''\right) - \frac{d^n}{dx^n}(xy') - m^2 y^n = 0$$

which, on use of the Leibniz rule, gives

$$(1-x^2)y^{n+2} - 2xny^{n+1} - n(n-1)y^n - xy^{n+1} - ny^n + m^2 y^n = 0,$$

or

$$(1-x^2)y^{n+2} - (2n+1)xy^{n+1} + (m^2 - n^2)y^n = 0.$$

Set $x = 0$. Then

$$y^{n+2}(0) = (n^2 - m^2)y^n(0).$$

But $y(0) = 0$; $y'(0) = m$ therefore coefficients of all even powers of x are zero. It also follows that $y'''(0) = m(1-m^2)$; $y^v(0) = m(1-m^2)(9-m^2)$ so that

$$y(x) = mx + \frac{m(1-m^2)}{3!}x^3 + \frac{m(1-m^2)(9-m^2)}{5!}x^5 + ...$$

Radius of convergence R is

$$R = \lim_{n\to\infty}\left(\frac{y^n(0)(n+2)!}{y^{n+2}(0)n!}\right) = \lim_{n\to\infty}\left(\frac{(n+2)(n+1)}{n^2-m^2}\right) = 1.$$

11. Let $y = \tan x$ then

$$\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

Take n-th derivative and use Leibniz rule:

$$y^{n+1} = \frac{d^n}{dx^n}y^2 = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k y}{dx^k}\frac{d^{n-k}y}{dx^{n-k}}$$

With $n = 2$ it follows that

$$y''' = yy'' + 2y'y' + yy''$$

so that, given $y(0) = 0,\ y'(0) = 1$, then $y'''(0) = 2$. Also, with $n = 4$,

$$y^v = yy^{iv} + 4y'y''' + 6y''y'' + 4y'''y' + yy^{iv}$$

so that $y^v(0) = 16$. Thus,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$$

As a check, use series expansions as in Sheet 2, question 4(f).

12. On use of the Taylor series formula,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$
$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...$$

Therefore,

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right).$$

Radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{a_{2n-1}}{a_{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2n-1}\right| = 1.$$

Take $x = 1/4$ so that $\frac{1+x}{1-x} = 5/3$ then

$$\log(5/3) = 2\left(\frac{1}{4} + \frac{1}{3}\left(\frac{1}{4}\right)^3 + \frac{1}{5}\left(\frac{1}{4}\right)^5 + \ldots\right) = 0.5108.$$

13. From Taylor expansions

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f^{iv}(a) + \ldots$$
$$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2!}f''(a) - \frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f^{iv}(a) + \ldots$$

Subtracting these two equations and rearranging yields

$$\frac{f(a+h) - f(a-h)}{2h} = f'(a) + \frac{h^2}{6}f'''(a)$$

while adding them and rearranging yields

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a) + \frac{h^2}{12}f^{iv}(a).$$

This gives the required results.

Now let $f(x) = \sin x$ and $h = \pi/12$. Then, according to the above approximations,

$$f'(\pi/4) \approx \frac{\sin(\pi/4 + \pi/12) - \sin(\pi/4 - \pi/12)}{\pi/6} = 0.6991$$

while exact value is $\cos(\pi/4) = 1/\sqrt{2} = 0.7071$. On the other hand

$$f''(\pi/4) \approx \frac{\sin(\pi/4 + \pi/12) - 2/\sqrt{2} + \sin(\pi/4 - \pi/12)}{(\pi/12)^2} = -0.7031$$

while exact value is $-\sin(\pi/4) = -1/\sqrt{2} = -0.7071$.